A model-theoretic analysis of geodesic equations in negative curvature

Rémi Jaoui

University of Notre Dame

Manchester Logic seminar

- **Geometric stability theory**: the geometry of definable sets in stable theories. More concretely, the geometry of strongly minimal sets definable in a given stable theory T. (Shelah, Zilber, Hrushovski, Pillay, ...)
- **Algebraic integrability of differential equations**: the study of the algebraic and transcendence properties of the solutions of algebraic differential equations. (Liouville, Jacobi, Painlevé, Poincaré,...)

Aim of my talk: describe some interaction between these two subjects around **differentially closed fields** and discuss examples coming from classical mechanics (geodesic motions on Riemannian manifolds).

Plan of my talk

(1) Strongly minimal sets in $\mathbf{DCF}_0$.

(2) Painlevé-irreducibility.

(3) Invariant foliations and invariant webs.

We work in the ambient theory $\mathbf{DCF}_0$: unless otherwise stated, types, definable sets and all notions of model theory are relative to the theory $\mathbf{DCF}_0$.

Theorem (Shelah, 73')

Let κ be an uncountable cardinal. There are 2^κ models of the theory $\mathbf{DCF}_0$ pairwise non isomorphic.

- a version of Morley's theorem: a complete theory is $\aleph_1$-categorical if and only if it is almost strongly minimal: there exists a model M and a strongly minimal set D such that $M \subset \mathrm{acl}(D)$.
- in contrast, $\mathbf{DCF}_0$ is ω-stable and multidimensional: the family of differential equations

$$y' = f(y) \text{ where } f(x) \in \mathbb{C}(x).$$

contains "many" pairwise orthogonal strongly minimal sets.

Definition

Two definable strongly minimal sets D_1 and D_2 are non-orthogonal if there is a definable strongly minimal set D_3 and definable finite-to-one maps f_1 and f_2

$$f_1 : D_3 \to D_1 \text{ and } f_2 : D_3 \to D_2$$

Higher-dimensional examples

In the same paper, Shelah conjectures that for every natural number $n \in \mathbb{N}$, there are new strongly minimal sets of order n:

$$y^{(n)} = f(y, y', y'', y^{n-1})$$

leading to even more multidimensionality.

New ideas originated from the work of Zilber in the 80's concerning totally categorical theories:

Observation

Let D be a strongly minimal set. The (model-theoretic) algebraic closure satisfies the ***exchange property****: for any set $A \subset D$ and $x, y \in D$*

$$x \in acl(A, y) \setminus acl(A) \Rightarrow y \in acl(A, x).$$

The pair (D, acl) is called a **combinatorial pregeometry**. To an arbitrary subset B of D, one can associate a dimension:

$$\dim(B) = max\{\mid B_0 \mid \text{with } B_0 \subset acl(B) \text{ acl-independent set}\}$$

Theorem (Zilber, 80's)

*Let D is a strongly minimal set definable in a ω-categorical theory. Then after a possible extension of parameters, D is **modular**: for any acl-closed subsets $A, B \subset D$:*

$$dim(A \cup B) = dim(A) + dim(B) - dim(A \cap B).$$

- In an ω-categorical theory, the algebraic closure is **locally finite**: the algebraic closure of a finite set is always finite.
- Zilber obtains a full classification of ω-categorical strongly minimal sets and deduce among other things:

  Any totally categorical theory is pseudofinite and finitely axiomatizable among infinite structures (quasi-finite axiomatizability).
- Such definable sets are called **locally modular**. When no extension of the parameters is required, we say D is **modular**.

Definition

A strongly minimal set D is disintegrated if for any subset $A \subset D$

$$acl(A) = \bigcup_{a \in A} acl(a).$$

Zilber's theorem: Let D be a strongly minimal set in an ω-categorical theory (in a saturated model).

(i) D is **disintegrated**: D is a definable finite cover of a definable set D_0 with trivial induced structure.

(ii) D is **purely modular**: D is a definable finite cover of a definable set D_0 such that there exists a bijection:

$$D_0 \to \mathbb{P}(V) \text{ where V is an } \mathbb{F}_q\text{-vector space of inf. dim.}$$

which send the definable structure to (an extension by constants) of the linear structure on $\mathbb{P}(V)$.

(iii) D is **locally modular non modular**: similar than (ii) with projective geometry replaced by affine geometry over a finite field.

Let $(E) : y^{(n)} = f(y, y', y'', y^{n-1})$ be any algebraic differential equation **of order n** $\geq$ **2**. We denote by D the associated definable set of $\mathbf{DCF}_0$.

Theorem (Hrushovski, 90's)

(i) *If D is strongly minimal then it is modular.*

(ii) *Let A be a non-isotrivial simple Abelian variety. There exists a differential equation defining a strongly minimal set $D_A \subset A$ containing all the torsion points of A.*

(iii) *D_A is modular and non-disintegrated: for every closed subvariety X of A^n*

$$\overline{X \cap D_A^n}^{Zar} \text{ is a union of cosets of subgroups of } A^n.$$

(iv) *Any non-disintegrated strongly minimal set of order $n \geq 2$ is (up to a generically finite to finite correspondence) of the form above.*

[Freitag-Scanlon 15', Casale-Freitag-Nagloo '19] describe other strongly minimal sets in $\mathbf{DCF}_0$ which carry remarkable **disintegrated non ω-categorical structures** coming from arithmetic geometry.

Let M be a smooth and connected algebraic subset of the Euclidean space $\mathbb{R}^N$.

Definition

A geodesic of M is a real-analytic curve $\gamma : I \to M$ such that:

- γ is parametrized by its arc-length that is with constant speed $= 1$.
- At every time $t \in I$, the acceleration $\gamma''(t)$ (seen as a vector in $\mathbb{R}^N$) is orthogonal to the tangent space $T_{\gamma(t)}M$ of M at $\gamma(t)$.

.

The geodesics of M are the analytic solutions of a common algebraic differential equation called the **geodesic equation of** M and denoted $Geo(M)$.

- If $M = \mathbb{R}^m$ then the geodesics are the lines.
- Abstract point view on geodesics on M: they are the trajectories of free (or undisturbed) particles moving on the curved space M.

  A body moving on a level surface will continue in the same direction at a constant speed unless disturbed. (Galileo's inertia principle)

Main Theorem

Let M be a smooth and connected algebraic subset of the Euclidean space $\mathbb{R}^N$. $Geo(M)$ denotes the geodesic equation of M.

Theorem

*Assume that M is **a compact surface** with everywhere **negative curvature**. Then the generic type of the algebraic differential equation $\mathrm{Geo}(M)$ (rel. to $\mathbf{DCF}_0$) is minimal and disintegrated.*

- The definable set D defined by $Geo(M)$ is $\mathbb{R}$-definable set of **$\mathbf{DCF}_0$ of order (or algebraic dimension)** 3. The theorem is a description of D modulo a smaller $\mathbb{R}$-definable set D_0 :

$$D \sim D' \text{ if and only if } D \triangle D' \text{ is small}$$

  that is contained in the set of solutions of a differential equation of order 2.
- The intersection of all D' with $D' \sim D$ defines a complete type $p \in S(\mathbb{R})$ called **the generic type of** D.
- A weaker notion of minimality adapted to types: a type p is minimal if all forking extensions are algebraic.

The formalism of geometric stability theory can be developed at the level of minimal types instead of strongly minimal sets.

Painlevé 1895: *Leçons sur la théorie analytique des équations différentielles: professées à Stockholm*

A question: Given an algebraic differential equation, when is it possible to write down the solutions of (E) using only rational functions and classical transcendental functions such as exponentials and elliptic functions?

- A differential field of **classical functions** is a (differential) subfield of $(\mathcal{M}(U), \frac{d}{dt})$ for some connected analytic open set $U \subset \mathbb{C}$

$$K_0 = \mathbb{C}(t) \subset K_1 \subset \ldots K_{n-1} \subset K_n = K$$

  obtained from the field $\mathbb{C}(t)$ of **rational functions** using repetitively a set $(\mathcal{P})$ of permissible operations assumed to be **classical integrations**.
- We say that the general solution of (E) can be expressed using only classical functions if the generic type of (E) is realized in a differential field of classical functions.

Everything can be made differential algebraic (without fixing a differential embedding into a field of meromorphic functions)

The set $(\mathcal{P})$ of permissible operations

(P1): solving an algebraic equation: $K_i \subset K_{i+1}$ is an algebraic extension.
(P2): solving a linear differential equation: K_{i+1} is generated over K_i by solutions of

$$Y' = AY \text{ for some matrix A } \in M_n(K_i)$$

(P3): solving an isoconstant abelian differential equation: Let Γ be a lattice such that $\mathbb{C}^n/\Gamma$ is an Abelian variety. K_{i+1} is generated over K_i by analytic functions of the form

$$\phi \circ \pi \circ (f_1, \ldots, f_n)$$

where $f_1, \ldots, f_n \in K_i$, ϕ is a meromorphic function on the Abelian variety $\mathbb{C}^n/\Gamma$ and π is the projection.

Proposition (over countable differential fields K of $\mathbb{C}(t)$)

Let (E) be an algebraic differential equation over K.
The generic type of (E) is analyzable in the constants if and only if the general solution of (E) can be expressed using only classical functions.

In short, a definable set D is analyzable in the constants if any minimal type living on D^{eq} is non-orthogonal to the constants.

Consider the differential equation:

$$(E_1) : y' = \frac{y}{y+1}$$

- Shelah (73') proved that (E_1) is not solvable by classical functions. It is the first step of the proof of the multidimensionality of $\mathbf{DCF_0}$.
- Rosenlicht (74') entirely solved the problem of solvability by classical functions for autonomous differential equations of order one:

$$y' = f(y) \text{ with } f \in \mathbb{C}(X).$$

Higher dimensional phenomena: Consider

$$(E_2) : z'' = \frac{z'z}{z+z'} + \frac{(z')^2}{z}$$

- (E_2) is also not solvable by classical functions but (up to a classical integration) its resolution can be reduced to the integration of (E_1) since:

$$(E_2) \iff \begin{cases} y' = \frac{y}{y+1} \\ z' = yz \end{cases} .$$

- Painlevé calls such second order equation **reducible**.

Let (E) be an algebraic differential equation (E) **of order** $n \geq 2$

Definition

We say that (E) is *Painlevé-irreducible* if the generic solution of (E) can not be realized in a differential field obtained from the field of rational functions with operations $(P1), (P2), (P3)$ and

(P4): solving any algebraic differential equation **of order** $<$ **n**.

Proposition (over countable differential subfields of $\mathbb{C}(t)$)

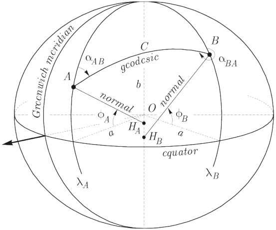
The following are equivalent:

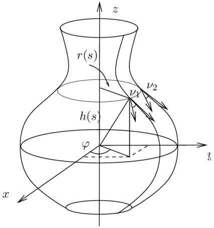
- *the differential equation (E) is Painlevé-irreducible,*
- *the generic type of (E) in* $\mathbf{DCF}_0$ *is minimal.*

The equivalence between these two formalisms goes back to Pillay '97 (on superstable differential fields) and Pillay-Nagloo '11 (on Painlevé equations).

Geodesic equations II

- **Smooth quadrics** (Jacobi): the geodesic equation of a smooth quadric surface $M \subset \mathbb{R}^3$ is solvable by classical functions.
  This means that the generic type of $Geo(M)$ is analyzable in the constants.
- **Surfaces of revolution:** the geodesic equation of a "general" surface of revolution $M \subset \mathbb{R}^3$ is an intermediate case: it is not Painlevé-irreducible and it is not solvable by classical functions.





Main Theorem: The geodesic equation of a compact surface with negative curvature is always Painlevé-irreducible.

Until the rest of the talk, we fix M a compact Riemannian surface with negative curvature and we denote by $N = SM$ the sphere bundle of M.

- points of N are pairs (x, u) where $x \in M$ and $u \in T_xM$ is a unit vector tangent to M. So N is a compact manifold of dimension 3.
- For every $(x, u) \in N$, there is a unique geodesic

  $$\gamma_{(x,u)} : \mathbb{R} \to N$$

  such that $\gamma_{(x,u)}(0) = (x, u)$.

Definition

The geodesic flow is the 1-parameter subgroup of analytic diffeomorphism of N given by letting all the geodesics evolve for a time t:

$$\begin{cases} \mathbb{R} \longrightarrow \mathrm{Diff}(N) \\ t \longrightarrow \phi_t : (x, u) \mapsto \gamma_{(x,u)}(t) \end{cases}$$

The main theorem relies on results of Anosov ('69) describing the "chaotic" dynamical properties of this one-parameter subgroup under the assumption of negative curvature.

Let $(N, (\phi_t)_{t\in\mathbb{R}})$ be the geodesic flow of M. For every $x \in N$, we consider three subsets of T_xN:

- the set $E_s(x)$ of "contracting" vectors:

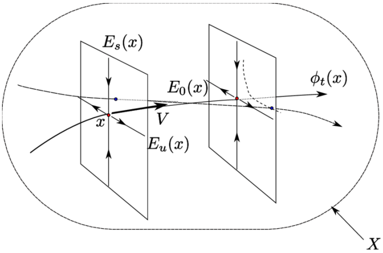
$$w \in T_xN \text{ with } \parallel d\phi_t(w) \parallel \overset{t\mapsto\infty}{\longrightarrow} 0$$

- the set $E_u(x)$ of "expanding" vectors:

$$w \in T_xN \text{ with } \parallel d\phi_t(w) \parallel \overset{t\mapsto-\infty}{\longrightarrow} 0$$

- the set $E_0(x)$ of "uniformly bounded" vectors $w \in T_xN$ such that

$$w \in T_xN \text{ with } t \mapsto \parallel d\phi_t(w) \parallel \text{ is uniformly bounded.}$$



(a) $E_s(x), E_u(x)$ and $E_0(x)$ are three **transversal lines** of T_xN.
(b) The lines $E_s(x), E_u(x)$ and $E_0(x)$ **vary continuously** with $x \in N$.
(c) $E_s(x), E_u(x)$ and $E_0(x)$ are **invariant**:

$$d\phi_t(E_{s,u,0}(x)) = E_{s,u,0}(\phi_t(x))$$

Let $\mathcal{F} = E_s, E_u$ or E_0 be any continuous field of line (a continuous foliation) on the manifold N.

Definition

The leaf of $\mathcal{F}$ through $x \in N$ is the real-analytic curve obtained on N obtained from x by always following the direction $\mathcal{F}_y \subset T_y$ prescribed by $\mathcal{F}$.

- The leaf of E_0 though $x \in N$ is the geodesic passing through $x_0 \in N$.
- The leaf $W_s(x)$ of E_s through $x \in N$ is the set of points with **the same asymptotic future than** x_0:

$$W_s(x) = \{y \in N \text{ such that } d(\phi_t(x), \phi_t(y)) \to 0 \text{ when } t \mapsto \infty\}.$$

- The leaf $W_u(x)$ of E_u through $x \in N$ is the set of points with **the same asymptotic past than** x:

$$W_u(x) = \{y \in N \text{ such that } d(\phi_t(x), \phi_t(y)) \to 0 \text{ when } t \mapsto -\infty\}.$$

Theorem (Anosov, Plante)

For every $x \in N$, the leaves $W_s(x)$ and $W_u(x)$ are dense in N.

Let X be a 3-dimensional smooth complex algebraic variety. We denote by T_X the tangent space of X and $\mathbb{P}(T_X)$ the projectivization.

$$\begin{tikzcd} T_X \setminus X \arrow[r] \arrow[d] & \mathbb{P}(T_X) \arrow[dl] \\ X & \end{tikzcd}$$

Definition

An **algebraic web on** X is a closed subvariety of $W \subset \mathbb{P}(T_X)$ such that every irreducible component of W dominates X and

$$\pi_{|W} : W \to X \text{ is gen. finite with generic fibres of cardinal } \dim(X) = 3.$$

- The singular locus of W is the set Z of points of X such that $\pi_{|W}$ is ramified or admits an infinite fibre over x.
- Outside of the singular locus, the fibre of $\pi_{|W}$ over x is a set of three lines

$$l_1(x), l_2(x), l_3(x) \in \mathbb{P}(T_{X,x})$$

The set of lines $x \mapsto \{l_1(x), l_2(x), l_3(x)\}$ vary algebraically with $x \in X \setminus Z$

Let (E) be an algebraic differential equation of dimension 3 defined over $\mathbb{R}$. We consider:

- The space $X(\mathbb{C})$ of (complex) initial conditions. It is the set of complex points of a (smooth) algebraic variety X defined over $\mathbb{R}$.
- The space $X(\mathbb{R})$ of real initial conditions and $(\phi_t)_{t\in\mathbb{R}}$ the real-analytic flow of (E) acting on $X(\mathbb{R})$.

Observation (real case)

*Assume that X is smooth and $X(\mathbb{R})$ Zariski-dense in X. Then there exists an **algebraic vector field** v_E on $\mathbb{P}(T_X)$ such that:*
*If $W \subset \mathbb{P}(T_X)$ is **an algebraic web tangent to** v_E with singular locus Z then:*

(i) *The set $Z(\mathbb{R})$ of real singularities of W is invariant under $(\phi_t)_{t\in\mathbb{R}}$.*

(ii) *On the dense open set $U(\mathbb{R}) = X(\mathbb{R}) \setminus Z(\mathbb{R})$, W defines a **smooth** real analytic frame of the complexified tangent space $T_{U(\mathbb{R})} \otimes \mathbb{C}$ which is invariant:*

$$d\phi_t(W_x) = W_{\phi_t(x)} \text{ for all } t \in \mathbb{R} \text{ and } x \in U(\mathbb{R}).$$

Such an algebraic web is called **an invariant web** of (E).

Theorem

Let (E) be a (real or complex) algebraic differential equation. Assume that the set of complex initial conditions of (E) is a smooth complex algebraic variety $X = X(\mathbb{C})$ of dimension 3. Assume that:

(i) *The differential equation (E) is **not solvable by classical functions**.*

(ii) *If $\mathcal{F}$ is an **invariant foliation** on X of rank $r = 1, 2$ then $\mathcal{F}$ has at least a Zariski-dense leaf.*

(iii) *If W is **an invariant algebraic web on X** then W has at least a Zariski-dense leaf.*

Then the generic type of (E) is minimal and disintegrated. So (E) is Painlevé-irreducible.

In (ii), algebraic foliations of rank one (resp rank two) are rational sections:

$$\sigma : X \dashrightarrow \mathbb{P}(T_X) \text{ (resp. } \mathbb{P}(T_X^*))$$

It is called an invariant foliation if the Zariski-closure of the image of σ is tangent to the vector field v_E.

Thank you for your attention!