Disintegrated differential equations and mixing Anosov flows

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A bit of history

- At the end of the XIX century, Picard and Vessiot develop a Galois theory for homogeneous linear differential equations.
  At the same period, Poincaré publishes the three volumes of "Les méthodes nouvelles de la mécanique celeste".
- $\sim$ 1900: In his thesis, Drach studies the broader class of algebraic differential equations whose complete solution can be algebraically parametrized, knowing a fundamental system of solutions $y_1, \ldots y_n$:

$$(*) \qquad y = f(c_1, \ldots c_n, y_1, \ldots y_n) \text{ where } c_1, \ldots, c_p \text{ are constants.}$$

  However, Painlevé finds inaccuracies in Drach's thesis.
- 1980's: Conjoint developments of differential algebra and geometric stability theory provide a new framework to study algebraic differential equations.
- Striking example: The work of Umemura (1987) and Nagloo-Pillay (2011) on Painlevé transcendents.

Naïve question

For a "highly non-integrable" algebraic differential equation, what is the nature of the algebraic relations shared by its solutions?

Autonomous differential equations

A differential equation is autonomous if it is defined without any explicit reference to the time t.

- Syntactic form : $(y^{(3)})^4 + 5y.y^{(2)} + (y')^5 = 0$.
- Explicit geometric form: a pair (X, v) where X is a (smooth) algebraic variety X over some field k endowed with a vector field v.

Definition (Closed invariant subvarieties)

Let (X, v) be a differential equation and Z be a closed subvariety of X. TFAE:

(i) The closed subvariety Z can be written as the Zariski-closure in X of an analytic solution $\gamma : \mathbb{D} \longrightarrow X(\mathbb{C})^{an}$.

(ii) The locally closed subvariety Z_{reg} is tangent to the vector field v on X.

(iii) The sheaf of ideals $\mathcal{I}_Z$ is invariant by the derivation induced by v on X.

For a differential equation (X, v) and $n \geq 1$, we denote by $\mathcal{I}_n = \mathcal{I}_n(X, v)$, the set of closed irreducible invariant subvarieties of $(X, v)^n$.

Disintegrated differential equations

Definition

The differential equation (X, v) is called *disintegrated* if for each $n \geq 3$, every $Z \in \mathcal{I}_n(X, v)$ can be written as an irreducible component of

$$\bigcap_{1 \leq i \neq j \leq n} \pi_{i,j}^{-1}(Z_{i,j}) \text{ where } Z_{i,j} \in \mathcal{I}_2(X, v) \text{ and } \pi_{i,j} : X^n \longrightarrow X^2.$$

- (Hrushovski-Itai, 2003) Let C be a smooth projective curve of genus ≥ 2 such that $\mathrm{Jac}(C)$ is a simple Abelian variety and let ω be a global 1-form on C. The differential equation

$$(E) : \omega(\frac{dy}{dt}) = 1 \text{ satisfies } \mathcal{I}_n(E) \text{ is finite for every } n \in \mathbb{N}.$$

- (Freitag-Scanlon, 2014) Let (E) be the order 3 differential equation (E) over $\mathbb{Q}$ satisfied by the j function and its $\mathrm{Gl}_2(\mathbb{C})$-conjugates. Then:
  - The differential equation (E) is disintegrated.
  - The set $\mathcal{I}_2(E)$ is a countable set of finite-to-finite correspondences.

Trichotomy Theorem of $\mathbf{DCF}_0$

Theorem (Hrushovski-Sokolovic, 1996)

Let $(\mathcal{U}, \delta_{\mathcal{U}})$ be a differentially closed field. The following description of minimal types with parameters in $(\mathcal{U}, \delta_{\mathcal{U}})$ holds:

(i) *If p is non-locally modular, then p is non-orthogonal to the generic type of the fields of constants.*

(ii) *If p is locally modular non disintegrated, then p is non-orthogonal to the generic type of the "Manin's Kernel" associated to a simple Abelian variety A over $\mathcal{U}$, which does not descend to the constants.*

(iii) *The type p is a minimal disintegrated type.*

Question: For a "highly non-integrable" differential equation, what are the possible minimal types involved in the semi-minimal analysis of its generic type?

Main setting

Let (X, v) be a smooth complex variety X endowed with a vector field v.

Theorem (Cauchy's Theorem)

For every point $p \in X(\mathbb{C})$ and for an open disk $\mathbb{D} \subset \mathbb{C}$ sufficiently small, there is a unique analytic curve

$$\gamma_p : \mathbb{D} \longrightarrow X(\mathbb{C})^{an}$$

tangent to the vector field v and satisfying $\gamma_p(0) = p$.

We want to think as the analytic solutions of (X, v) as the dynamical system on $X(\mathbb{C})^{an}$ defined by:

$$\begin{cases} \mathbb{D} \subset \mathbb{C} & \longrightarrow \mathrm{Aut}(X(\mathbb{C})^{an}) \\ t & \longrightarrow p \mapsto \gamma_p(t) \end{cases}$$

Two basic obstructions:

- Explosion in finite time : even for t small enough, the local solutions $\gamma_p(t)$ may not be defined for all $p \in X(\mathbb{C})^{an}$ simultaneously (solved by compactness).
- Monodromy : There is no global determination of the logarithm on $\mathbb{C}^*$ (consider the differential equation $x' = \frac{1}{x}$).

Main setting (Case of real numbers)

Main setting

Let (X, v) be an absolutely irreducible variety endowed with a vector field defined over the field $\mathbb{R}$ of real numbers. Assume $(\dagger)$:

(i) *The set $X(\mathbb{R})$ is contained in the regular locus of X.*

(ii) *The set $X(\mathbb{R})^{an}$ is compact.*

(iii) *The set $X(\mathbb{R})$ is Zariski-dense in X (here, equivalent to $X(\mathbb{R}) \neq \emptyset$).*

Under $(\dagger)$, we have two different objects associated to the same differential equation:

- The real-analytic flow $(M, (\phi_t)_{t \in \mathbb{R}})$ acting on the set $M = X(\mathbb{R})^{an}$ of real-initial conditions.
- The generic type of the differential (X, v), which is a stationary type of the theory $\mathbf{DCF_0}$.

Anosov flows

Let M be a compact real-analytic manifold, v an analytic vector field on M and $(\phi_t)_{t\in\mathbb{R}}$ its real-analytic flow. Choose a metric $||.||$ on the tangent bundle TM of M and define for $x \in M$:

$$E^{ss}_x = \{v \in TM_x \ , \ ||d\phi_t(v)|| \overset{t\mapsto+\infty}{\longrightarrow} 0\}$$

$$E^{su}_x = \{v \in TM_x \ , \ ||d\phi_t(v)|| \overset{t\mapsto-\infty}{\longrightarrow} 0\}.$$

Definition

The flow $(M, (\phi_t)_{t\in\mathbb{R}})$ is an Anosov flow if:

(i) E^{ss} and E^{su} are (non-trivial) continuous sub-bundles of TM and the convergence is uniformly exponentially fast.

(ii) Transversality : $TM = E^{ss} \oplus \mathbb{R}.v \oplus E^{su}$.

"*Local product structure*": Let $\epsilon > 0$ and $p_1, p_2 \in M$. If $(M, (\phi_t)_{t\in\mathbb{R}})$ is a (topologically transitive) Anosov flow, there exists a point q and $t_1 < t_2$ such that:

- The point q follows the orbit of p_1 (up to ϵ) for $t < t_1$.
- The point q follows the orbit of p_2 (up to ϵ) for $t > t_2$.

Examples of Anosov flows

- Classical examples: Let $\Sigma \subset \mathbb{R}^n$ be a smooth compact (non-empty) algebraic subset of dimension ≥ 2.

Theorem (Anosov, 1969)

If Σ has negative curvature, then the system of differential equations describing the movement of a particle, constrained to move without friction on Σ is described by a vector field with a real-analytic Anosov flow.

- Robust notion: Let M is a compact (real)-manifold. The set of smooth vector fields which define a compact Anosov flow on M is open in $\mathcal{C}^\infty(M, TM)$ endowed with the $\mathcal{C}^1$ topology.
- However, having a Anosov flow is not a "typical" property for a smooth vector field on M. For example, KAM Theorem prevents small perturbations of completely integrable Hamiltonian systems to satisfy such a global hyperbolic behavior.

Main Theorem - Model theoretic form

Theorem (J.)

Let (X, v) be an absolutely irreducible D-variety over $\mathbb{R}$. Assume that the real-analytification $X(\mathbb{R})^{an}$ of X is compact non-empty and contained in the regular locus of X.
If the real-analytic flow $(X(\mathbb{R})^{an}, (\phi_t)_{t\in\mathbb{R}})$ is a mixing Anosov flow of dimension 3, then exactly one of the two following cases holds:

- (i) *Either the generic type of (X, v) is minimal and disintegrated.*
- (ii) *Or there exists a strictly disintegrated type r of order 1 over $\mathbb{R}$ such that the generic type of (X, v) and $r^{(3)}$ are interalgebraic over $\mathbb{R}$.*

- (Eberlein, 1973) Geodesic flows of compact manifolds of negative curvature are always mixing Anosov flows.
- Anosov alternative implies that a (topologically transitive) non-mixing Anosov flow can always be written as the suspension of a diffeomorphism over a constant roof function (hence, they are of a very special kind).

Main Theorem - Geometric form

For $n \geq 2$, denote by $\mathcal{I}_n^{gen}$ the set of closed invariant subvarieties of $(X, v)^n$ which project generically on all the factors.
A differential equation (X, v) is *generically disintegrated* if for every $n \geq 3$, every $Z \in \mathcal{I}_n^{gen}$ can be written as an irreducible component (projecting generically on each factor) of:

$$\bigcap_{1 \leq i \neq j \leq n} \pi_{i,j}^{-1}(Z_{i,j}) \text{ where } Z_{i,j} \in \mathcal{I}_2^{gen} \text{ and } \pi_{i,j} : X^n \longrightarrow X^2.$$

Theorem (J.)

*Let (X, v) be an absolutely irreducible D-variety over $\mathbb{R}$ satisfying $(\dagger)$.
If the real-analytic flow $(X(\mathbb{R})^{an}, (\phi_t)_{t \in \mathbb{R}})$ is a mixing Anosov flow of dimension 3, then the D-variety (X, v) is generically disintegrated.
Moreover, one of the following two cases holds:*

(i) *Either $\mathcal{I}_2^{gen}$ contains only generically finite-to-finite correspondences and the generic type of (X, v) is minimal.*

(ii) *Or $\mathcal{I}_2^{gen}$ contains at least a closed subvariety of $X \times X$ of codimension 1. Moreover, in that case, $\mathcal{I}_2^{gen}$ is a finite set.*

Possible strengthening of the main theorem

- More precise description of the algebraic relations shared by the solutions of (X, v) are related to more precise description of the set $\mathcal{I}_2^{gen}$.
  In particular, for an algebraically presented Anosov flow of dimension 3, it is natural to expect that the set $\mathcal{I}_2^{gen}$ is both finite and consists of finite-to-finite correspondences.
- (Pereira-Couthinho, 2005) For a "very generic" rational vector field" on a smooth projective variety X , every non-generic solution of (X, v) is stationary at a singular point of v.
  Under that hypothesis, the distinction between $\mathcal{I}_n^{gen}$ and $\mathcal{I}_n$ collapses.
- Question: Similar statements for other "typical" dynamical behavior of smooth vector fields on a compact manifold ?
  A particularly interesting case is that of small perturbations of algebraically integrable Hamiltonian systems (such as the three-body problem).

Model-theoretic core of the proof

Theorem (J.)

Let $(\mathcal{U}, \delta_{\mathcal{U}})$ be a differentially closed field, C_0 be a subfield of the field $\mathcal{C}$ of constants of $(\mathcal{U}, \delta_{\mathcal{U}})$ and $p \in S(C_0)$ be a stationary type.
If p is a type of order 3 which is semi-minimal and orthogonal to the constants, then one of the two following cases holds:

(i) *The type p is minimal and disintegrated.*

(ii) *There exists a strictly disintegrated type $q \in S(A)$ of order 1 such that $q^{(3)}$ and p are interalgebraic over C_0.*

- The type p is non-orthogonal to a minimal type $r \in S(K)$ (defined after a possible extension of the parameters). The proof consists in applying Hrushovski-Sokolovic Theorem to the type r.
- Hence, it suffices to prove that the generic type of a differential equation of dimension 3 (satisfying (†)) with a mixing Anosov flow is both orthogonal to the constants and semi-minimal.

Orthogonality to the constants and semi-minimality

Let (X, v) an absolutely irreducible variety endowed with a vector field. A rational dominant map $f : (X, v) \dashrightarrow (Y, w)$ towards another D-variety (Y, w) over k is a rational map $f : X \dashrightarrow Y$ such that:

$$df(v) = w.$$

Proposition (Orthogonality to the constants)

The generic type of (X, v) is orthogonal to the constants if and only if for every $n \in \mathbb{N}$, there are no rational dominant map $f : (X, v)^n \dashrightarrow (\mathbb{A}^1, 0)$.

Proposition (Semi-minimality)

Assume that they are no dominant rational map $f : (X, v) \dashrightarrow (Y, w)$ towards another D-variety (Y, w) unless $\dim(Y) = 0$ or $\dim(Y) = \dim(X)$ (equivalently, unless Y is a point or f is generically finite).
Then, the generic type of (X, v) is semi-minimal.

Weakly mixing dynamics

Let $(M, (\phi_t)_{t\in\mathbb{R}})$ be a metric flow. For every (non-empty) open subsets $U, V \subset M$, set

$$N(U, V) = \{t \in \mathbb{R} \mid \phi_t(U) \cap V \neq \emptyset\}.$$

The flow $(M, (\phi_t)_{t\in\mathbb{R}})$ is called topologically transitive if $N(U, V) \neq \emptyset$ whenever $U, V \subset M$ are non-empty open subsets.

Lemma (Weakly mixing flows)

Let $(M, (\phi_t)_{t\in\mathbb{R}})$ be a metric flow. TFAE:

(i) *The flow $(M, (\phi_t)_{t\in\mathbb{R}}) \times (M, (\phi_t)_{t\in\mathbb{R}})$ is topologically transitive.*

(ii) *For every topologically transitive metric flow $(N, (\psi_t)_{t\in\mathbb{R}})$, the flow $(N, (\psi_t)_{t\in\mathbb{R}}) \times (M, (\phi_t)_{t\in\mathbb{R}})$ is topologically transitive.*

(iii) *The set*

$$\{N(U, V) \mid U, V \text{ non-empty open subsets of } M\} \subset \mathcal{P}(\mathbb{R})$$

form a filter basis on $\mathbb{R}$.

Invariant foliations on a smooth D-variety (X, v)

Definition

An invariant foliation $\mathcal{F}$ on (X, v) is an involutive saturated coherent subsheaf of the sheaf $\Theta_{X/k}$ of vector fields on X, invariant by the Lie-derivative $\mathcal{L}_v$ of the vector field v.

Proposition

Let $f : (X, v) \dashrightarrow (Y, w)$ be a rational dominant map defined and smooth on a dense open set U. Then:

(1) *The tangent foliation $T_f = \mathrm{Ker}(df_{|U})$ of $T_{U/k}$ extends uniquely to a (possibly singular) foliation $\mathcal{F}_f$ on X of rank $\dim(X) - \dim(Y)$.*

(2) *The foliation $\mathcal{F}_f$ is invariant on (X, v).*

(3) *The singular locus of $\mathcal{F}_f$ is a closed invariant subvariety of X.*

Assume that $(\dagger)$ holds and that the real-analytic flow $(M, (\phi_t)_{t\in\mathbb{R}})$ of (X, v) is a mixing Anosov flow of dimension 3.
Using the continuous sub-bundles E^{ss} and E^{su} of TM, we proved that any (algebraic) invariant foliation $\mathcal{F}$ on (X, v) of positive rank has Zariski-dense leaves.