On the solutions of planar algebraic vector fields

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Abstract

Theorem (J., preprint 19')

Let $\mathcal{V}_d$ denotes the family of complex algebraic vector fields of degree $\leq d$ on the complex plane $\mathbb{C}^2$. For $d \geq 3$, for almost all vector fields $v \in \mathcal{V}_d$, the differential equation associated to the vector field v is strongly minimal and disintegrated (has trivial forking geometry).

This theorem describes the structure (in the sense of model-theory) of the set of solutions in a differentially closed field of a planar vector field chosen randomly among algebraic vector fields of degree d where $d \geq 3$.

Plan of the talk:

(1) Describe the content of the conclusion of the theorem above in differential-algebraic terms.

(2) Explain how model-theory is used in the proof of the theorem.

(3) Describe the linearization technique used in the proof of the theorem.

Vocabulary and notation

We consider differential equations of the form

$$(E) : \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \text{where } f(x,y), g(x,y) \in \mathbb{C}[x,y].$$

associated to planar algebraic vector fields $v(x,y) = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$.
The vector field v induces a derivation of $\mathbb{C}[x,y]$ that extends uniquely to $\mathbb{C}(x,y)$ defined by

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Vocabulary:

- A rational integral of (E) is a rational function $f \in \mathbb{C}(x,y)$ such that
$$\delta_v(f) = df(v) = 0.$$
- A complex invariant curve C for (E) is an affine algebraic curve invariant under the (local) flow of the the vector field v. If $C := (f = 0)$, this can be expressed algebraically as:
$$\delta_v(f) = df(v) = hf \text{ for some } h \in \mathbb{C}[x,y].$$
- A generic solution of (E) is a solution of (E) in a differential field extension of $(\mathbb{C}, 0)$ which is not a zero of v and not contained in any complex (invariant) curve.

First example: Hamiltonian systems with one degree of freedom

Consider a Hamiltonian $H(p,q) = \frac{1}{2}p^2 + V(q)$ and the associated Hamiltonian differential equation:

$$\begin{cases} \dot{q} = p \\ \dot{p} = -V'(q) \end{cases} \qquad \text{described by the vector field } v_H = p\frac{\partial}{\partial q} - V'(q)\frac{\partial}{\partial p}.$$

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- The Hamiltonian $H : (\mathbb{C}^2, v_H) \to (\mathbb{C}, 0)$ is a rational integral of v_H so the the integration of X_H can be reduced to the integration of the one-dimensional differential equation:

$$(E_h) : \frac{1}{2}(\frac{dq}{dt})^2 + V(q) = h \text{ defined over } (\mathbb{C}(h), 0).$$

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- Classically, the system is known to be (analytically) completely integrable. Using the method of separation of variables, one can associate to (E_h) the indefinite integral:

$$(*) : dt = \int \frac{dq}{\sqrt{2h - 2V(q)}}.$$

The general solution of (E_h) is given by $q(t) = F_h^{-1}(t + C)$ where F_h is an antiderivative of $(*)$.

Semi-minimal analysis of Hamiltonian systems with one degree of freedom

We distinguish three cases according to the degree of the potential $V(q)$:

- If $\deg(V(q)) = 2$ then (E_h) admits a generic solution in a Picard-Vessiot extension of $\mathbb{C}(h)^{alg}$.
  Classically, after a change of coordinates, $(*)$ can be reduced to:

$$dt = \int \frac{dq}{\sqrt{1-(\omega q)^2}} \text{ so } t = \frac{1}{\omega}\arcsin(\omega q) + C.$$

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- If $\deg(V(q)) = 3, 4$, then (E_h) admits a generic solution in a strongly normal extension of $\mathbb{C}(h)^{alg}$ but (in general) not in a Picard Vessiot extension of $\mathbb{C}(h)^{alg}$.
  Classically after a change of coordinates, $(*)$ can be reduced to
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- For generic values of $V(q)$ with $\deg(V(q)) \geq 5$, (E_h) does not admit a generic solution in a strongly normal extension of $\mathbb{C}(h)^{alg}$.

[(Rosenlicht '74); (Hrushovski,Itai 03'); (Noordman, van der Put, Top 11')]

Second example: Pullbacks by logarithmic derivative

Consider the family of planar algebraic vector fields:

$$(E_f) : \begin{cases} \dot{y} = xy \\ \dot{x} = f(x) \end{cases} \qquad \text{with } f(x) \in \mathbb{C}(x).$$

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- If $f(x) = x$ then (E_f) does not admit a generic solution in a strongly normal extension of $(\mathbb{C}, 0)$ **but** does admit one in an iterated PV-extension of $(\mathbb{C}, 0)$ of the form:

$$(\mathbb{C}, 0) \subset (K_1, \delta_1) \subset (K_2, \delta_2)$$

where $(K_1, \delta_1)|(k, 0)$ and $(K_2, \delta_2)|(K_1, \delta_1)$ are PV-extensions.

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- For generic values of $f(x)$ of degree ≥ 3, then (E_f) does not admit a generic solution in an iterated strongly normal extension of $(\mathbb{C}, 0)$ **but** does in a "mixed extension" of the form:

$$(\mathbb{C}, 0) \subset (K_1, \delta_1) \subset (K_2, \delta_2)$$

  where $(K_1, \delta_1)|(k, 0)$ is not strongly normal of transcendence degree one and $(K_2, \delta_2)|(K_1, \delta_1)$ is strongly normal.

[Jin-Moosa '19] gives necessary and sufficient conditions on $f(x)$ to distinguish these three cases.

Main result

Theorem (J. 19')

Let $\mathcal{V}_d$ denotes the family of complex algebraic vector fields of degree $\leq d$ on the complex plane $\mathbb{C}^2$. For $d \geq 3$, for almost all vector fields $v \in \mathcal{V}_d$,

- ***Minimality**: (E_v) does not admit any non constant solution in a differential extension of the form:*

$$(\mathbb{C}, 0) \subset (K_1, \delta_1) \subset (K_2, \delta_2) \subset \cdots \subset (K_n, \delta_n)$$

  where each of the steps in the tower above is either
  - *an algebraic extension*
  - *a strongly normal extension,*
  - *or a differential field extension of transcendence degree one.*
- ***Disintegration:** if $(x_1, y_1), \ldots, (x_n, y_n)$ are n solutions of (E_v), then $(x_1, y_1), \ldots, (x_n, y_n)$ are algebraically independent over $\mathbb{C}$ **unless***

$$P(x_j, y_j, x_i, y_i) = 0$$

  for some $i \neq j$ and a polynomial $P \neq 0$.

Comments on minimality

- generic/non-constant solutions.

Theorem (Landis-Petrovskii, 58')

Let $\mathcal{V}_d$ denotes the family of complex algebraic vector fields of degree $\leq d$ on the complex plane $\mathbb{C}^2$. For $d \geq 2$, for almost all vector fields $v \in \mathcal{V}_d$, any analytic curve on X tangent to v is either stationary at a zero of v or Zariski-dense in $\mathbb{C}^2$.

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- In the language of model theory, (i) can be restated as: *the solutions of (E_v) in a differentially closed field form a strongly minimal definable set.* This uses:

irreducibility (Nishioka-Umemura) $\Leftrightarrow$ p_v is minimal $\Leftrightarrow$ p_v is strongly minimal.

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- What happens for $d = 1, 2$? The correct picture for $d = 2$ is still unclear. It boils down to:

Question

Does there exist a complex quadratic planar vector field without rational integral and whose generic solutions do not lie in the algebraic closure of a strongly normal extension of $(\mathbb{C}, 0)$?

Comments on disintegration

It is natural to expect that generic vector fields of sufficiently high degree satisfy a stronger (and more explicit) version of the disintegration property.

Definition

We say that $\mathrm{Dis}(n, d)$ holds if for almost all vector fields $v \in \mathcal{V}_d$, n solutions $(x_1, y_1), \ldots, (x_n, y_n)$ of (E_v) are algebraically independent over $\mathbb{C}$ **unless**:

- one of them is a constant solution,
- or two of them are equal.

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- Landis-Petrovskii theorem states that $Dis(1, d)$ holds when $d \geq 2$.
- A specalization argument shows that $Dis(n, d) \Rightarrow Dis(n, d+1)$
- Disintegration implies that $Dis(2, d) \Rightarrow Dis(n, d)$ for every $d \geq 3$ and every $n \geq 2$.

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Question

Does there exists $d \geq 2$, such that $\mathrm{Dis}(2, d)$ holds? Is it possible to compute such a d explicitly?

Strategy of the proof

Consider a differential equation

$$(E) : \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \text{where } f(x,y), g(x,y) \in \mathbb{C}[x,y].$$

We want to identify sufficient conditions to ensure that the set of solutions of (E) in a differentially closed field is strongly minimal and disintegrated.

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- Does (E) admit a non-trivial rational integral?
- Does (E) admit a generic solution in (the algebraic closure of) a strongly normal extension of $(\mathbb{C}, 0)$?

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- Does (E) admit a non-trivial rational integral?
- Does (E) admit a generic solution in (the algebraic closure of) a strongly normal extension of $(\mathbb{C}, 0)$?
- Can (E) be reduced by a change of coordinates

  $$u = u(x,y), v = v(x,y)$$

  (and more generally, a finite to finite correspondence) to a system of differential equations in triangular form

  $$\begin{cases} h(u,u',v) = 0 \\ g(v,v') = 0 \end{cases} \quad ?$$

Rational and algebraic factors

Let X be a complex algebraic surface endowed with a vector field v.

Definition

A rational factor of (X, v) of dimension one is a triple (C, v_C, ϕ) where

- C is a complex algebraic curve and v_C a vector field on C.
- $\phi : X \dashrightarrow C$ is a dominant rational morphism satisfying $d\phi(v) = v_C$.

An algebraic factor of (X, v) of dimension one is a diagram

$$\begin{tikzcd} (X', v') \arrow[r, dotted, "\phi"] \arrow[d, dotted, "\rho"] & (C, v_C) \\ (X, v) & \end{tikzcd} \quad \text{where} \quad \begin{cases} \rho \text{ is dominant generically finite,} \\ v' \text{ is the extension of } v \text{ to } X', \\ (C, v_C, \phi) \text{ is a rational factor of } (X', v'). \end{cases}$$

Observation: A system $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ can be made triangular after a generically finite to finite correspondance if and only if $(\mathbb{A}^2, f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y})$ admits an algebraic factor of dimension one.

Consequence of the Hrushovski-Sokolovic Trichotomy in $\mathbf{DCF}_0$

Theorem

Consider a differential equation

$$(E): \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

satisfying:

(i) (E) *does not admit non trivial rational integrals.*

(ii) (E) *does not admit a generic solution in the algebraic closure of a strongly normal extension of* $(\mathbb{C}, 0)$.

(iii) $(\mathbb{A}^2, f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y})$ *does not admit an algebraic factor of dimension one.*

Then the generic type p_E *of* (E) *is strongly minimal and disintegrated.*

General planar algebraic vector fields

Let $\mathcal{V}_d$ denotes the family of complex algebraic vector fields of degree $\leq d$ on the complex plane $\mathbb{C}^2$.

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- I showed in "Orthogonalité aux constants pour les équations différentielles autonomes" (18') that (ii) holds for for almost all vector fields of $\mathcal{V}_d$ for $d \geq 3$.

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Theorem (J. 19')

Let (X, v) be a smooth irreducible complex algebraic surface endowed with a vector field. Assume that there exists a zero $p \in X(\mathbb{C})$ of v such that:

(i) ***Hyperbolicity and non-resonance:*** *the eigenvalues λ, μ of the linear part of v at p are non zero and satisfy $\lambda/\mu \notin \mathbb{Q}_+ \cup \mathbb{R}_-$.*

(ii) ***No algebraic separatrix:*** *the zero p is not contained in any complex invariant algebraic curve C.*

Then (X, v) does not admit any algebraic factor of dimension one.

Main protagonist: the D-module $(\Omega^1_X, \mathcal{L}_v)$

Let X be a complex algebraic surface endowed with a vector field v. For every open set $U \subset X$, the vector field v induces

- a derivation δ_v on $\mathcal{O}_X(U)$ defined by $\delta_v(f) = df(v)$.
- a D-module structure on $\Omega^1_X(U)$ over the differential ring $(\mathcal{O}_X(U), \delta_v)$ determined by:

$$\mathcal{L}_v(df) = d(\delta_v(f)) \text{ for every } f \in \mathcal{O}_X(U).$$

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When U varies in X, the D-modules $(\Omega^1_X(U), \mathcal{L}_v)$ define a sheaf of D-modules on X over the sheaf of differential rings $(\mathcal{O}_X, \delta_v)$.

- The stalk $(\Omega^1_X, \mathcal{L}_v)_\eta$ at the generic η of X which is a D-module over the differential field $(\mathbb{C}(X), \delta_v)$.
- The stalk $(\Omega^1_X, \mathcal{L}_v)_p$ at an hyperbolic and non-resonant zero p of v.

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After applying Poincaré's linearization theorem around p, we can compute:

$$(\mathcal{O}_X, \delta_v)^{an}_p \simeq (\mathbb{C}\{x, y\}, \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y})$$

and in the basis (dx, dy), the $(\Omega^1_X, \mathcal{L}_v)^{an}_p$ is described by

$$\mathcal{L}_v \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

From rational factors of (X, v) to D-submodules of $\Omega^1_{X,\eta}$

If $\phi : X \dashrightarrow C$ is a dominant rational morphism, it is generically smooth so we get an exact sequence (over the generic point η of X):

$$0 \to \phi^*\Omega^1_{C,\eta} \overset{d\phi}{\to} \Omega^1_{X,\eta} \to \Omega^1_{X/C,\eta} \to 0 \text{ of } \mathbb{C}(X)\text{-vector spaces.}$$

The image $\Omega(\phi)$ of $\phi^*\Omega^1_{C,\eta}$ in $\Omega^1_{X,\eta}$ is a $\mathbb{C}(X)$ vector-subspace of dim. 1.

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Lemma

The correspondence

$$\mathrm{Lin} : \; (X, v) \xrightarrow{\phi} (C, v_C) \mapsto \Omega(\phi)$$

sends *one-dimensional rational factors of* (X, v) ***to*** *one dimensional D-submodules of* $(\Omega^1_X, \mathcal{L}_v)_\eta$*, which are algebraically integrable.*

Here, we say that a D-submodule M of $(\Omega^1_X, \mathcal{L}_v)_\eta$ of dimension one is algebraically integrable if equivalently:

- it is generated by a one-form of the form df for some $f \in \mathbb{C}$.
- the dual $M^\vee \subset \Theta_X$ is generated by a rational vector field on X with a non-trivial rational integral.

From D-submodules of $(\Omega^1_X, \mathcal{L}_v)_\eta$ to invariant foliations on (X, v)

To study the D-submodules F of $(\Omega^1_X, \mathcal{L}_v)_\eta$ from the point of view of an hyperbolic singularity p of $X(\mathbb{C})$, we extend them (in a canonical way) into D-coherent sheaves $\mathcal{F}$ of $(\Omega^1_{X,\eta}, \mathcal{L}_v)$ and consider the stalk $\mathcal{F}_p$.

Proposition (Saturation)

(i) *Any one dimensional $\mathbb{C}(X)$-subvector space F of $\Omega^1_{X,\eta}$ extends uniquely into an invertible subsheaf $\mathcal{F}$ of Ω^1_X such that*

$$\Omega^1_X/\mathcal{F} \text{ is torsion free.}$$

(ii) *If F is a D-submodule of $(\Omega^1_X, \mathcal{L}_v)_\eta$ then $\mathcal{F}_p$ is a D-submodule of $(\Omega^1_X, \mathcal{L}_v)_p$.*

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To study the D-submodules F of $(\Omega^1_X, \mathcal{L}_v)_\eta$ from the point of view of an hyperbolic singularity p of $X(\mathbb{C})$, we extend them (in a canonical way) into D-coherent sheaves $\mathcal{F}$ of $(\Omega^1_{X,\eta}, \mathcal{L}_v)$ and consider the stalk $\mathcal{F}_p$.

Proposition (Saturation)

(i) *Any one dimensional $\mathbb{C}(X)$-subvector space F of $\Omega^1_{X,\eta}$ extends uniquely into an invertible subsheaf $\mathcal{F}$ of Ω^1_X such that*

$$\Omega^1_X/\mathcal{F} \textit{ is torsion free.}$$

(ii) *If F is a D-submodule of $(\Omega^1_X, \mathcal{L}_v)_\eta$ then $\mathcal{F}_p$ is a D-submodule of $(\Omega^1_X, \mathcal{L}_v)_p$.*

Such an invertible subsheaf $\mathcal{F}$ is called *a (possibly singular) invariant foliation* on (X, v). If ω is a local generator of $\mathcal{F}$ around p then we have two cases:

- the 1-form induced by ω on the tangent space $T_{X,p}$ at p is identically zero. We say that the foliation is singular at p.
- Otherwise, the kernel is a one-dimensional subspace of $T_{X,p}$ that is a line at p. We say that the foliation is called regular at p.

Rational factors: sketch of proof

Let X be a complex algebraic variety, v a vector field on X and $p \in X(\mathbb{C})$ an hyperbolic and non-resonant zero of v. Consider $\pi : (X, v) \dashrightarrow (C, v_C)$ a rational factor of dimension one.

Step 1 **Linearization:** rational factor $\pi \rightsquigarrow$ a D-submodule F_π of $\Omega^1_{X,\eta}$ which algebraically integrable.

Step 2 **Extension:** a D-submodule F_π of $\Omega^1_{X,\eta} \rightsquigarrow$ an algebraically integrable foliation $\mathcal{F}_\pi$ on X invariant by $\mathcal{L}_v$.

Step 3 **Analytic coordinates at p:** Choose analytic coordinates (x, y) around p such that in this new coordinates, v is equal to its linear part.

Step 4 **Local analysis of linear vector fields:**

Lemma

Let $v(x, y) = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$ be a linear vector field with $\lambda/\mu \notin \mathbb{Q}$ and let $\mathcal{F}$ an analytic foliation invariant by v defined on a neighborhood of 0.

- *If $\mathcal{F}$ is non singular at 0 then $\mathcal{F}$ is either the horizontal or vertical foliation.*
- *If $\mathcal{F}$ is singular at 0 then $\mathcal{F}$ is linear in the same coordinates (x, y) with the same eigenvectors than (the linear part of v).*

Step 5 **Conclusion:** Use that $\mathcal{F}_\pi$ is algebraically integrable to conclude that p is contained in a complex invariant curve.

A word about (non-rational) algebraic factors

Instead of a rational factor, we can start with an algebraic factor:

$$\begin{tikzcd} (X', v') \arrow[r, dotted, "\phi"] \arrow[d, dotted, "\rho"] & (C, v_C) \\ (X, v) & \end{tikzcd} \quad \text{where} \begin{cases} \rho \text{ is dominant generically finite,} \\ v' \text{ is the extension of } v \text{ to } X', \\ (C, v_C, \phi) \text{ is a rational factor of } (X', v'). \end{cases}$$

- Up to extending X', we can assume that $k(X) \subset k(X')$ is a finite Galois extension of fields with Galois group G.
- From the rational factor (C, v_C, ϕ), we still get an algebraically integrable invariant foliation $\mathcal{F}$ on X'.

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- Up to extending X', we can assume that $k(X) \subset k(X')$ is a finite Galois extension of fields with Galois group G.
- From the rational factor (C, v_C, ϕ), we still get an algebraically integrable invariant foliation $\mathcal{F}$ on X'.
- Although the foliation $\mathcal{F}$ does not generally descend on X', we can kill the action of G by considering

$$x \in X(\mathbb{C}) \mapsto \bigcup_{\sigma \in G} \sigma(F_q) \subset T_p X \text{ where } q = \rho(x).$$

which associates to a generic point x of X, a set of $\sharp G$ lines in $T_p X$.

A word about (non-rational) algebraic factors

Note that if V is a vector space of dimension two and $\omega_1, \ldots, \omega_n \in V^*$ are linear form defining lines $l_1, \ldots, l_k \subset$ then:

$$l_1 \cup \ldots \cup l_k = \{v \in V, \Omega(v) = 0\} \text{ where } \Omega = \prod_{i=1}^{k} \omega_i \in \mathrm{Sym}^k(V^*).$$

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Definition

A k-web of foliations on X is an invertible subsheaf $\mathcal{W}$ of $\mathrm{Sym}^k(\Omega^1_X)$ such that $\mathrm{Sym}^k(\Omega^1_X)/\mathcal{W}$ is torsion-free.

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Sketch of proof for algebraic factors:

- an algebraic factor of $(X, v) \rightsquigarrow$ an invariant k-web $\mathcal{W}$ on (X, v) satisfying certain algebraicity properties.
- Let p denote the hyperbolic and non-resonant zero of v. Distinguish whether $\mathcal{W}$ is locally decomposable at p (in an analytic neighborhood) as a product of k analytic foliations or not.

References:

- Corps differentiels et flots géodésiques I: Orthogonalité aux constantes pour les équations différentielles autonomes (arXiv:1612.06222).
- Generic planar algebraic vector fields are disintegrated (arXiv:1905.09429).

Thank you for your attention!