Disintegrated differential equations

Rémi Jaoui

University of Waterloo

2 August 2018

Differential Algebra and Related Topics IX

Algebraic relations for the solutions of an ODE

- Naïve problem: Given an algebraic differential equation (E), describe all the possible algebraic relations for its solutions and their derivatives.
- For example, if (E) is an autonomous differential equation of the form

$$y^{(2)} = F(y, y') \text{ with } F \in \mathbb{C}[X_1, X_2].$$

For every $p \in \mathbb{N}$, we want to describe the set $\overline{\mathcal{I}_p(E)}$ of polynomials (or system of polynomials) $Q \in \mathbb{C}[X_1, X_2, \ldots, X_{2p}]$ such that:

$$Q(y_1, y_1', \ldots, y_p, y_p') = 0 \text{ for some solutions } y_1, \ldots, y_p \text{ of (E) .}$$

- Similar questions were already raised at the end of the XIX century by Picard and Vessiot (for linear differential equations), Poincaré (for the three-body problem) and by Painlevé and Drach.
- In my talk, I will explain some results on the behavior of the sequence $(\overline{\mathcal{I}_p(E)})_{p \in \mathbb{N}}$ when p varies for some "highly non-integrable" differential equations, using tools from model-theory.

Autonomous differential equations

A differential equation is autonomous if it is defined without any explicit reference to the time t.

- Syntactic form : $(y^{(3)})^4 + 5y.y^{(2)} + (y')^5 = 0$.
- Explicit geometric form: a pair (X, v) where X is a (smooth) algebraic variety X over some field k endowed with a vector field v.

Definition (Closed invariant subvarieties)

Let (X, v) be a differential equation and Z be a closed subvariety of X. TFAE:

(i) *The locally closed subvariety Z_{reg} is tangent to the vector field v on X.*

(ii) *The sheaf of ideals $\mathcal{I}_Z$ is invariant by the derivation induced by v on X.*

For a differential equation (X, v) and $p \geq 1$, we denote by $\mathcal{I}_p(X, v)$, the set of closed irreducible invariant subvarieties of $(X, v)^p$.
Given p (analytic) solutions $y_1, \ldots, y_p$ of the differential equation (X, v), the Zariski-closure of $(y_1, \ldots, y_p)$ belongs to $\mathcal{I}_p(X, v)$.

Disintegrated differential equations

The collection $\mathcal{L} = \{\mathcal{I}_p(X, v) \mid p \in \mathbb{N}\}$ is closed under various operations:

- irreducible components of intersections of elements in $\mathcal{I}_p(X, v)$.
- Zariski-closure of projections under $\pi : X^p \rightarrow X^q$ with $p \leq q$.
- irreducible components of pull-backs under $\pi : X^p \rightarrow X^q$ with $p \leq q$.

Definition

The differential equation (X, v) is called disintegrated if for every $p \geq 3$, every $Z \in \mathcal{I}_p(X, v)$ can be written as an irreducible component of

$$\bigcap_{1 \leq i < j \leq p} \pi_{i,j}^{-1}(Z_{i,j}) \text{ where } Z_{i,j} \in \mathcal{I}_2(X, v) \text{ and } \pi_{i,j} : X^n \longrightarrow X^2.$$

The disintegration property (here, in a geometric form) is a structural property that plays a key role in various applications of model theory to concrete geometric settings. It was defined by Zilber for strongly minimal sets in order to isolate the ones that present "a lack of structure".

Some examples

- A linear differential equation is never disintegrated. More generally, any differential equation that admits a linear differential equation as a "rational factor" is not disintegrated.
- (Hrushovski-Itai, 2003) Let v be a vector field on a smooth quasi-projective curve C. The following are equivalent:
  - The differential equation (C, v) is disintegrated.
  - For every $p \in \mathbb{N}$, the set $\mathcal{I}_p(X, v)$ is finite.

  Moreover, for $C = \mathbb{A}^1$ a vector field v of degree $d \geq 3$ with algebraically independent coefficients is disintegrated.
- (Freitag-Scanlon, 2014) Let (E) be the order 3 differential equation (E) over $\mathbb{Q}$ satisfied by the j function and its $\mathrm{Gl}_2(\mathbb{C})$-conjugates.
  The differential equation (E) is disintegrated and the set $\mathcal{I}_2(E)$ is a countable set of finite-to-finite correspondences.

Together with the (non-autonomous) Painlevé equations with generic parameters (studied by Nagloo and Pillay), these differential equations and those that can be derived from them are the only ones known to satisfy this disintegration property.

Geodesic differential equation

Let $M \subset \mathbb{R}^n$ a smooth algebraic subset of the Euclidean space. The Euclidean metric g_0 on the Euclidean space induces a Riemannian metric g on M^{an}.

- The geodesic differential equation of $M \subset \mathbb{R}^n$ is the differential equation describing the movement of a particle constrained to move without friction on M.
- After writing $M = Y(\mathbb{R})$ for some quasi-projective variety Y (with Zariski-dense real points), this equation can be presented algebraically as the Hamiltonian vector field v_H on the symplectic variety $T_{Y/\mathbb{R}}$ associated to the Hamiltonian:

$$H(x, y) = \frac{1}{2} g_x(y, y).$$

- In fact, our results concern the unitary geodesic differential equation (when we fix the energy $E = E_0$), namely the fibres of the Hamiltonian fibration

$$H : (T_{Y/\mathbb{R}}, v_H) \longrightarrow (\mathbb{A}^1, 0).$$

In particular, if M has dimension n then the unitary geodesic differential equation of M lives in dimension $2n - 1$.

Disintegration in negative curvature

For $n \geq 2$, denote by $\mathcal{I}_n^{gen}(X, v)$ the set of closed irreducible invariant subvarieties of $(X, v)^n$ which project generically on all the factors.
A differential equation (X, v) is *generically disintegrated* if the system of algebraic relations $\{\mathcal{I}_n^{gen}(X, v) \mid n \in \mathbb{N}\}$ satisfies the disintegration property.

Theorem 1 (J.)

Let $M \subset \mathbb{R}^n$ be a smooth algebraic subset of the Euclidean space of dimension 2 and denote by (X, v) the unitary geodesic differential equation of M.
If M is a compact algebraic subset of (everywhere) negative curvature then the differential equation (X, v) is generically disintegrated.
Moreover, one of the following two cases holds:

(i) *Either $\mathcal{I}_2^{gen}(X, v)$ contains only generically finite-to-finite correspondences.*
(ii) *Or $\mathcal{I}_2^{gen}(X, v)$ contains at least a closed subvariety of $X \times X$ of codimension 1. Moreover, in that case, $\mathcal{I}_2^{gen}(X, v)$ is a finite set.*

Strategy of the proof: Describing the algebraic relations

(1) **Existence of a generically disintegrated factor** The proof of Theorem 1 starts by proving the existence of a rational factor (Y, w) of (X, v), which is generically disintegrated.

Definition

A rational factor of a differential equation (X, v) is a dominant rational map

$$\phi : (X, v) \dashrightarrow (Y, w)$$

towards another differential equation (Y, w) of positive dimension and such that $d\phi(v) = w$.

- This part of the proof is fairly general. It does not require any dimension assumption on X.
- We also prove the existence of a generically disintegrated factor for the vector fields on $\mathbb{A}^n$ of degree $d \geq 3$ with algebraically independent coefficients.
- We except a similar statement for certain small perturbations of completely integrable Hamiltonian system (such as the three-body problem).

Strategy of the proof II: Describing the algebraic relations

(2) **Factors of a differential equation of dimension 3.** The disintegration property is stable by generically finite-to-finite correspondences. Hence, to prove generic disintegration for (X, v), it suffices to show that

$$\pi : (X, v) \dashrightarrow (Y, w)$$

is generically finite (where (Y, w) is a generically disintegrated factor).

Theorem 2 (J.)

Let $M \subset \mathbb{R}^n$ be a smooth algebraic subset of the Euclidean space of dimension 2 and denote by (X, v) the unitary geodesic differential equation of M.
If M is a compact algebraic subset of (everywhere) negative curvature then (X, v) does not have any rational factors of dimension 1 or 2.

The proof of (2) relies on the "small dimension" assumptions (indeed, we assume in Theorem 1 and here that $\dim(X) = 3$).

(3) **Fine analysis of the set $\mathcal{I}_2^{gen}(X, v)$** of algebraic relations between two generic solutions of (X, v).

Orthogonality to the constants

Recall that if (X, v) is an absolutely irreducible differential equation, a rational integral of (X, v) is a rational function $f \in k(X)$ such that $v(f) = 0$. This rational integral is said to be non-constant if $f \notin k$.

Definition

An absolutely irreducible differential equation (X, v) is said to be orthogonal to the constants if for every $n \in \mathbb{N}$, the differential equation $(X, v)^n$ does not admit any non-constant rational integral.

- The terminology "orthogonal to the constants" comes here from model-theory. The equivalence between the model-theoretic notion and this geometric formulation is a theorem of my thesis.
- A linear differential equation is never orthogonal to the constants.
- A (generically) disintegrated differential equation is orthogonal to the constants. Conversely...

Orthogonality to the constants and disintegrated factors

Theorem 3 (J.)

Let (X, v) be an absolutely irreducible, autonomous differential equation. If (X, v) is orthogonal to the constants then (X, v) admits a generically disintegrated factor (of positive dimension)

$$\pi : (X, v) \dashrightarrow (Y, w).$$

- Theorem 3 is obtained through a decomposition procedure for (systems of) algebraic differential equations into simpler ones called semi-minimal analysis, characteristic of the methods of model-theory.
- The proof uses the full classification (up to a finite-to-finite correspondences) of minimal non-disintegrated differential equations given by Hrushovski-Sokolovic theorem.

Main setting (Case of real numbers)

Main setting

Let (X, v) be a differential equation defined over the field $\mathbb{R}$ of real numbers. Assume $(\dagger)$:

(i) *The set $X(\mathbb{R})$ is contained in the regular locus of X.*

(ii) *The set $X(\mathbb{R})^{an}$ is compact.*

(iii) *The set $X(\mathbb{R})$ is Zariski-dense in X (here, equivalent to $X(\mathbb{R}) \neq \emptyset$).*

Under $(\dagger)$, we can associate to the differential equation (X, v), its real-analytic flow $(M, (\phi_t)_{t\in\mathbb{R}})$

- It is an action of the additive group $(\mathbb{R}, +)$ on the set $M = X(\mathbb{R})^{an}$ of real-initial conditions.
- It is defined by

$$\begin{cases} \mathbb{R} & \longrightarrow \mathrm{Aut}(X(\mathbb{R})^{an}) \\ t & \longrightarrow p \mapsto \gamma_p(t) \end{cases}$$

  where γ_p is the unique solution of the real-analytification $(M, v_{\mathbb{R}}^{an})$ starting at the point p $(\gamma_p(0) = p)$.

Weakly mixing dynamics

Let $(M, (\phi_t)_{t\in\mathbb{R}})$ be a metric flow. For every (non-empty) open subsets $U, V \subset M$, set

$$N(U, V) = \{t \in \mathbb{R} \mid \phi_t(U) \cap V \neq \emptyset\}.$$

The flow $(M, (\phi_t)_{t\in\mathbb{R}})$ is called topologically transitive if $N(U, V) \neq \emptyset$ whenever $U, V \subset M$ are non-empty open subsets.

Lemma (Weakly mixing flows)

Let $(M, (\phi_t)_{t\in\mathbb{R}})$ be a metric flow. TFAE:

(i) *The flow $(M, (\phi_t)_{t\in\mathbb{R}}) \times (M, (\phi_t)_{t\in\mathbb{R}})$ is topologically transitive.*

(ii) *For every topologically transitive metric flow $(N, (\psi_t)_{t\in\mathbb{R}})$, the flow $(N, (\psi_t)_{t\in\mathbb{R}}) \times (M, (\phi_t)_{t\in\mathbb{R}})$ is topologically transitive.*

(iii) *The set*

$$\{N(U, V) \mid U, V \text{ non-empty open subsets of } M\} \subset \mathcal{P}(\mathbb{R})$$

form a filter basis on $\mathbb{R}$.

A dynamical criterion for orthogonality to the constants

Theorem 4 (J.)

Let (X, v) be an autonomous differential equation satisfying $(\dagger)$. Denote by $(M, (\phi_t)_{t\in\mathbb{R}})$ the real-analytic flow of (X, v). Then:

$$(M, (\phi_t)_{t\in\mathbb{R}}) \text{ is weakly mixing} \Longrightarrow (X, v) \text{ is orthogonal to the constants.}$$

- The weakly-mixing property for the geodesic differential equation of a compact Riemannian manifold in negative curvature follows from the work of Anosov in the 70's.
- The conclusion of Theorem 4 then implies the existence of a disintegrated factor (this is indeed the content of Theorem 3).

Thank you for your attention!