Strong minimality and invariant foliations

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Introduction

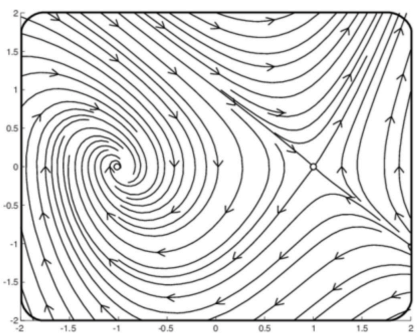
The purpose of my talk will be to describe an interaction between:

- the model theory of differential fields and more precisely, the model-theory of the "largest" differential fields: the differentially closed fields of char. 0 . (Shelah, Hrushovski, Pillay,...)
- the properties of algebraic integrability and algebraic independence of the solutions of algebraic differential equations. (Painlevé, Nishioka, Umemura,...)

and to study this interaction in certain families of differential equations using

- a description of the dynamic of the (real or complex) analytic solutions of the differential equations in this family.

For an order 2 equation $y'' = F(y, y')$ where $F \in \mathbb{R}[X, Y]$, this interaction will occur as:



The first-order structure $\mathcal{S}(E)$ formed by the solutions of (E) in a "universal" diff. field.

- algebraic integrability
- algebraic independence of the solutions of (E)

An example: geodesic equations of a real-algebraic Riemannian surface

The examples I will describe are three-dimensional systems of algebraic differential equations describing a geodesic movement on a compact Riemannian surface M.

Geodesic equations of a compact Riemannian surface: Consider $M \subset \mathbb{R}^n$ a regular, compact, connected real-algebraic subset of the Euclidean space endowed with the restriction of the Euclidean metric.

- We denote by $Geo(M)$ the (system of) differential equations describing the movement of unitary geodesics on M:

  an analytic curve $t \mapsto \gamma(t)$ on M is a solution of $Geo(M)$ if and only if it is the trajectory of a particle constrained to move without friction with constant speed $= 1$ along M

- We only consider the unitary geodesics so the space of initial conditions for $Geo(M)$ is the sphere bundle SM of M. The differential equation $Geo(M)$ can therefore be represented under the form

  $$Geo(M) : (SM, v) \text{ where } v \text{ is an (explicit) vector field on } SM.$$

  Since M is real-algebraic, both SM and v are also real-algebraic and $Geo(M)$ is a system of algebraic differential equations.

Problem: Describe the first-order logic structure of the set of solutions of $Geo(M)$ in a "universal" differential field from information on the Riemannian surface $(M, g_{|M})$.

Corollary

Let $M \subset \mathbb{R}^n$ be a real-algebraic regular compact surface embedded in the Euclidean space $\mathbb{R}^n$. Assume that

the restriction of the Euclidean metric to M has negative curvature at every point $p \in M$

and consider $t \mapsto \gamma(t)$ a "generic" (i.e. Zariski-dense in SM) unitary geodesic on M, then:

(A) *The geodesic $\gamma(t)$ is algebraically independent from all elliptic functions, solutions of linear differential equation over $\mathbb{C}(t)$, solutions of order 2 algebraic differential equations $F(y, y', y'') = 0$ with $F \in \mathbb{C}(t)[y_0, y_1, y_2]$:*

If $f_1(t), \ldots, f_r(t)$ are such analytic functions, the Zariski-closure of the analytic curve

$$t \mapsto (\gamma(t), \gamma'(t), t, f_1(t), \ldots, f_r(t)) \text{ in } SM \times \mathbb{A}^{r+1} \text{ is of the form } SM \times Z.$$

(B) *There are (at most) countably many algebraic subvarieties $(R_k \subset SM \times SM, i \in \mathbb{N})$ such that every n "generic" geodesics $\gamma_1, \ldots, \gamma_n$ of M are algebraically independent (over $\mathbb{C}$) i.e.*

$$t \mapsto (\gamma_1(t), \gamma_1'(t), \ldots, \gamma_n(t), \gamma_n'(t)) \text{ is Zariski-dense in } SM \times \ldots \times SM$$

unless for some $i \neq j$ and some $k \in \mathcal{I}$,

$$(\gamma_i(t), \gamma_i'(t), \gamma_j(t), \gamma_j'(t)) \in R_k.$$

I will describe how to gather all this information (and more) inside the description of a single object: the solution set $\mathcal{S}(Geo(M))$ in a "universal" differential field.

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Notation and conventions

- Autonomous differential equation: I will use two different representations for autonomous differential equations:

(X, v) where
X is a smooth alg. variety
endowed with a vector field v.

$$\rightleftarrows$$

$F(y, y', y'', \dots, y^{(n)}) = 0$
with $F \in \mathbb{C}[X_0, \dots, X_n]$
and $\frac{\partial F}{\partial X_n} \neq 0$.

For $\rightarrow$, choose a primitive element $f \in \mathbb{C}(X)$ i.e. such that

$$\mathbb{C} < f >= \mathbb{C}(f, \delta_v(f), \dots, \delta_v^n(f)) = \mathbb{C}(X).$$

For $\leftarrow$,

$$\begin{cases} X := (F(y_0, \dots, y_n) = 0 \wedge \frac{\partial F}{\partial y_n} \neq 0) \subset Jet^n(\mathbb{A}^1) \\ v = y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + \dots + y_n \frac{\partial}{\partial y_{n-1}} + \alpha(F) \frac{\partial}{\partial y_n}. \end{cases}$$

where $\alpha(F)$ is chosen such that v is tangent to X.

- We will be interested in "generic" solutions of the differential equation (X, v). Assuming that (X, v) does not admit any rational integral, they can be realized as

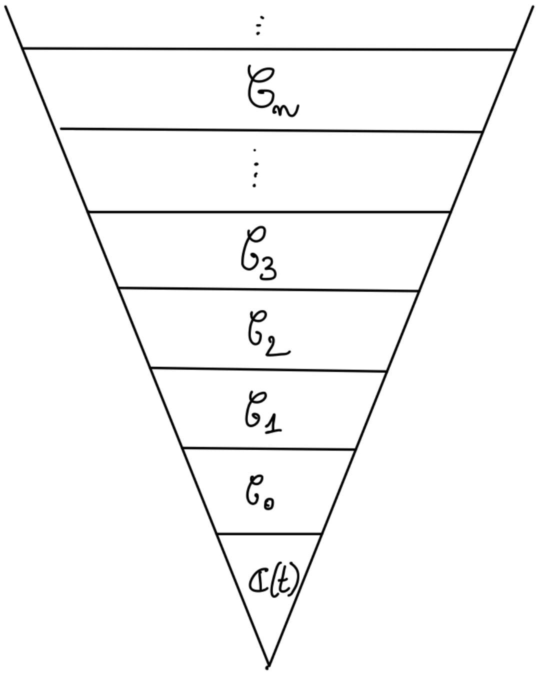
analytic curves $t \mapsto \gamma(t)$ on X solutions of (X, v) and Zariski-dense in X.

- I will also consider algebraic differential equation defined over $\mathbb{C}(t)$:

$$\pi : (X, v) \to (\mathbb{A}^1, \frac{d}{dt}) \quad \rightleftarrows \quad F(t, y', y'', \dots, y^{(n)}) = 0.$$

Painlevé's hierarchy of meromorphic functions

First goal: Describe the notion of algebraic integrability relevant to the understanding of the first order structure $\mathcal{S}(E)$. This notion of algebraic integrability has been identified by Painlevé and formalized in a modern language by Nishioka and Umemura.



The differential field $\mathcal{M}(U)$ of meromorphic functions on some open connected $U \subset \mathbb{C}$

- on the horizon, all the meromorphic functions which are differentially transcendental over $\mathbb{C}(t)$.
- as close as possible, the class $\mathcal{C}_0$ of "classical meromorphic functions" containing:
  - all the rational functions (in $\mathbb{C}(t)$).,
  - the exponential function, the logarithms, elliptic functions,
  - every meromorphic function that can be obtained from those using a small number of elementary integration procedures such as solving linear differential equation (of arbitrary order)
- every class $\mathcal{C}_k$ for $k \geq 1$,
  - contains all the solutions on U of algebraic differential equations of order $\leq k$.
  - is closed under the elementary integration procedures.

The classes $\mathcal{C}_k$ of meromorphic functions (Umemura)

Definition (Umemura)

A differential field $K \subset (\mathcal{M}(U), \frac{d}{dt})$ of meromorphic functions of the class $\mathcal{C}_k$ is a subdifferential field of $\mathcal{M}(U)$ that can be generated from the field $\mathbb{C}(t)$ of rational functions using a combinaison of the following operations:

(A1) Solving an *algebraic equation*.

(C1) Solving a *linear differential equation* (of arbitrary order):

$$y^{(r)} + b_{r-\mathbf{1}} y^{(r-\mathbf{1})} + \ldots + b_{\mathbf{1}} y = 0 \text{ where } b_{\mathbf{1}}, \ldots b_{r-\mathbf{1}} \text{ are already in } \mathcal{C}_k.$$

(C2) *Composing* meromorphic functions in the class $\mathcal{C}_k$ *with an Abelian function*.

(P_k) Solving *an algebraic differential equation* (non linear) of order $\leq k$:

$$G(y, y', \ldots, y^{(r)}) = 0 \text{ with } \mathbf{1} \leq \mathbf{r} \leq \mathbf{k} \text{ and the coefficients } G \text{ are already in } \mathcal{C}_k.$$

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The choice of the operations $(A1), (C1), (C2)$ can be justified using Kolchin's differential Galois theory: if (E) admits a Galois theory in the sense of Kolchin and G denotes its Galois group then:

- One can solve (E) using the operation $(C1)$ if the Galois group G is a linear algebraic group (it is Picard-Vessiot Galois theory).
- The operation $(C2)$ allows to solve the differential equation (E) if the Galois group is an Abelian variety.

The general problem of integration in my talk

Given an algebraic differential equation defined over $\mathbb{C}(t)$:

$$(E) : F(t, y, y', y'', \dots, y^{(n)}) = 0.$$

- Cauchy's theorem ensures the existence of some analytic function $f(t) \in \mathcal{M}(U)$ on some open disk $U \subset \mathbb{C}$ satisfying the differential equation (E).
- By definition, the function $f(t)$ belongs to $\mathcal{C}_r \setminus \mathcal{C}_{r-1}$ for some $r \leq n$.

A problem of integration: Given an algebraic differential equation (E) of order n over $\mathbb{C}(t)$, describe

(i) the combinations of the operations

$$(A1), (C1), (C2) \text{ and } (P_k) \text{ for } k = 1, \dots, n$$

producing a generic solution of (E).

(ii) the minimal class $\mathcal{C}_r$ in which live the generic solutions of (E).

Under some mild irreducibility assumptions on the space of initial conditions of (E), (i) and (ii) does not depend on the chosen generic solution (differential algebra).

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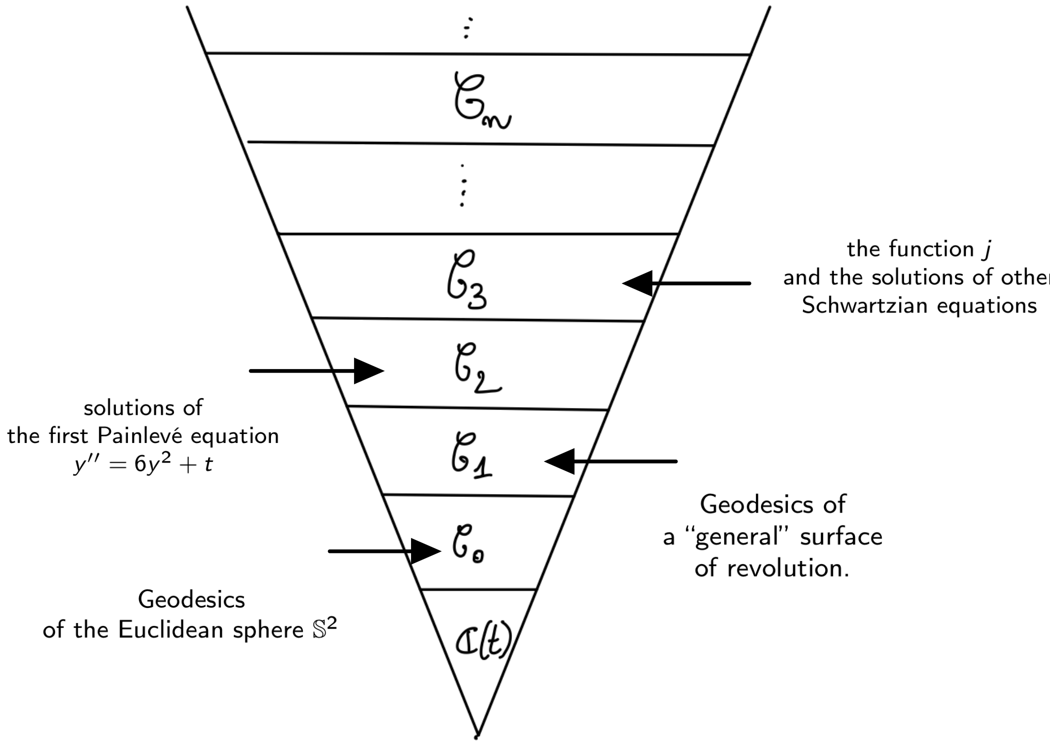
Definition

We will say that the generic solutions are new meromorphic functions (in the sense of Painlevé) if the generic solutions of (E) live in $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$.

This means that there are no combinations of $(A1), (C1), (C2)$ and (P_k) for $k = 1, \ldots, n-1$ producing a generic solution of (E).

Some examples



Questions: Where do the geodesics of a compact real-algebraic Riemannian surface with negative curvature lie in this classification? Why is it useful to study integration of algebraic differential equations under that form?

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Structure associated to an algebraic differential equation

Let (X, v) be an autonomous equation.

- For every non-empty open set $U \subset \mathbb{C}$, we can consider the meromorphic solutions of (X, v) on U as a subset of the set $X(\mathcal{M}(U))$ of all meromorphic curves from U to X,

$$\mathcal{S}_U(X, v) \subset X(\mathcal{M}(U))$$

- Using a logical limit (or amalgamation) procedure, we combine all the $\mathcal{S}_U(X, v)$ for various open sets $U \subset \mathbb{C}$ into a single "limit" object obtained by "taking the solutions" in a universal differential field $(\mathcal{U}, \delta)$, fixed once for all.

$$\mathcal{S}(X, v) \subset X(\mathcal{U}), \text{ where } (\mathcal{U}, \delta) \text{ is a "universal" differential field.}$$

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The limit object $\mathcal{S}(X, v)$ called the set of solutions of (X, v) is well-defined as a first-order structure.

(1) For every $n \geq 1$, $\mathcal{S}(X, v)^n$ is equipped with a Noetherian topology: the trace of the Zariski topology on $X(\mathcal{U})^n$ with basic closed sets of the form

$$Z(\mathcal{U}) \cap \mathcal{S}(X, v)^n \text{ for every closed complex subvariety } Z \text{ of } X^n.$$

(2) The $\emptyset$-definable subsets $\mathcal{S}(X, v)$ are obtained using logical connectives and quantifiers as usual (boolean combinations and projections).

We also have a similar construction for non-autonomous equations.

What did we gain going to the limit object?

From the point of view of logical complexity, the limit objects $\mathcal{S}(X, v)$ are "very tame" structures.

Definition

We say that a structure M is strongly minimal if every definable subset (with parameters) of M is either finite or cofinite.

Two heirs for the notion of strong minimality:

strongly minimal structures

o-minimal structures
(the structure supports at least a definable linear ordering)

finite rank structures
which can be decomposed into finitely many strongly minimal sets

What kind of structure are the $\mathcal{S}(X, v)$?

- They are stable structures : they (and their elementary extensions) do not interpret any infinite linear ordering.
- They have finite rank: they can be "decomposed" (or coordinatized) in terms of strongly minimal sets.

When is $\mathcal{S}(X, v)$ strongly minimal?

Answer: always when X is a curve (equivalently, for an order 1 differential equation). More surprisingly, it is also the case when X is higher dimensional but the solutions of (X, v) are new meromorphic functions.

Proposition (Pillay, Umemura)

Let (X, v) be an autonomous differential equation with $dim(X) = n > 1$. Assume that the structure $\mathcal{S}(X, v)$ is strongly minimal. Then

$$(*) : \textit{the generic solutions of } (X, v) \textit{ are new meromorphic functions of the class } \mathcal{C}_n.$$

Conversely, if $(*)$ is satisfied , the (type-definable) subset $\mathcal{S}(X, v)_{gen} \subset \mathcal{S}(X, v)$ consisting of generic solutions of (X, v) is minimal.

- The second condition is a weakening of the notion of strong minimality "localized around the generic solutions". It means that for every A-definable subset D of $\mathcal{S}(X, v)$ (with one variable):

$$\text{either } E_1 = D \cap \mathcal{S}(X, v)_{gen} \text{ or } (\mathcal{S}(X, v) \setminus D) \cap S(X, v)_{gen} \text{ is contained in } acl(A).$$

- I will now explain how to exploit this connection between new meromorphic functions (in the sense of Painlevé) and (strongly) minimal sets to describe the algebraic relations among new meromorphic functions.

An aside: Zilber trichotomy for o-minimal structures

$(\mathbb{R}, <)$	$(\mathbb{R}, 0, 1, +, -, \{\lambda_r r \in \mathbb{R}\}, <)$	$(\mathbb{R}, 0, 1, +, -, \times, <)$	
pure ordered set	semi-linear geometry	semi-algebraic geometry	expansions of real-algebraic geometry

Theorem (Peterzil-Starchenko, 97)

Let $\mathcal{R}$ be an $\aleph_1$-saturated model of an o-minimal theory T and $a \in \mathcal{R}$. Exactly one of the the following cases occurs:

(i) *a is a disintegrated point (or a trivial point): there does not exist an open interval I containing a and a continuous definable function*

$$f : I \times I \to M \text{ strictly monotone in both coordinates.}$$

(ii) *a is a modular (non disintegrated) point: there exists a convex neighborhood $\mathcal{V}$ a such that the structure induced by $\mathcal{R}$ on $\mathcal{V}$ is the structure of an ordered vector space over an ordered ring.*

(iii) *a is a non locally modular point: here exists a convex neighborhood $\mathcal{V}$ a such that the structure induced by $\mathcal{R}$ on $\mathcal{V}$ is an o-minimal expansion of real algebraic geometry.*

Example: Consider the structure $\mathcal{R}_{int} = (\mathbb{R}, 0, 1, +, -, \times_{|[-1,1]})$. The multiplication can be defined on every bounded subset of $\mathbb{R}$ but is not globally definable.

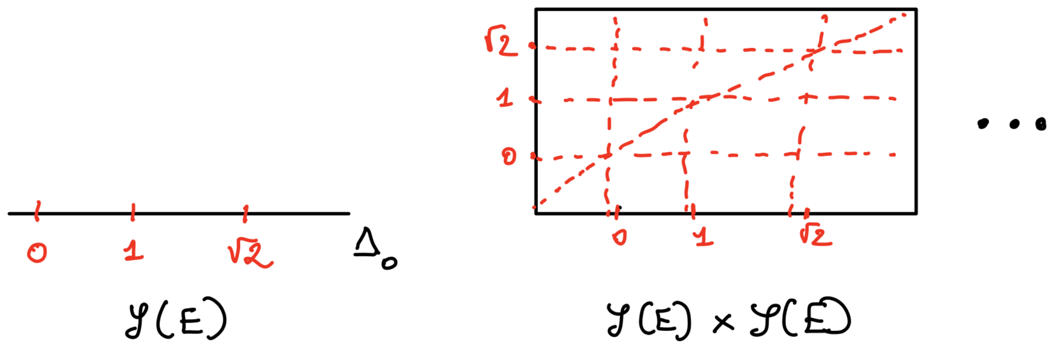
Order one equations and the classes $\mathcal{C}_0$ and $\mathcal{C}_1$

I will first describe the structure $\mathcal{S}(E)$ of an order 1 equation (E) whose generic solutions are new meromorphic functions of the class $\mathcal{C}_1$ (i.e live in $\mathcal{C}_1 \setminus \mathcal{C}_0$).

Example 1:

$$(E) : y' = y^2(y-1)(y-\sqrt{2})$$

The structure of the "limit object" $\mathcal{S}(E)$ is just a pure infinite set Δ_0 with three symbols of constants $\{0, 1, \sqrt{2}\}$ for the singularities of (E).



This means that the closed complex algebraic subvarieties Z of $\mathbb{A}^n$ always intersect $\mathcal{S}(E)$ in a trivial way: for every $Z \subset \mathbb{A}^n$

$$Z \cap \mathcal{S}(E)^n \text{ is a finite union of diagonals and slices such as } \{a\} \times \mathcal{S}(E)^{n-1}$$

where $a \in \{0, 1, \sqrt{2}\}$ is a singularity of (E).

Order one equations and the classes $\mathcal{C}_0$ and $\mathcal{C}_1$

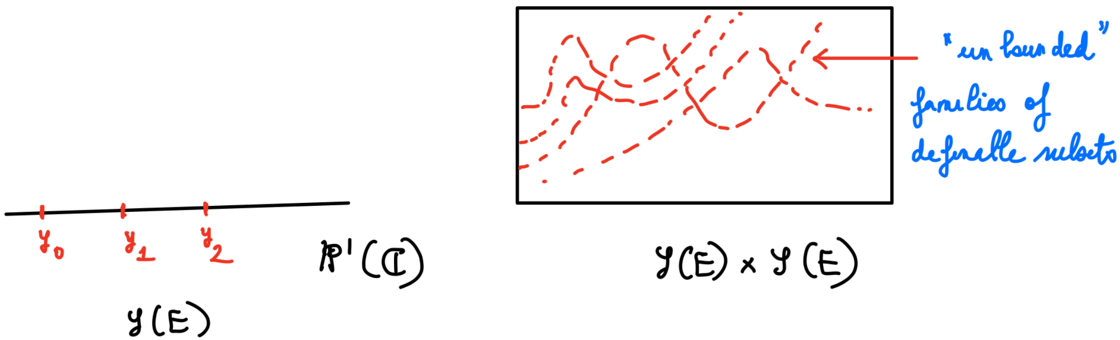
I now describe the structure $\mathcal{S}(E)$ of an order 1 equation (E) whose generic solutions are classical meromorphic functions from the class $\mathcal{C}_0$.

Example 2: A (non autonomous) Ricatti equation with rational coefficients $a(t), b(t), c(t) \in \mathbb{C}(t)$

$$(E) : y' = a(t)y^2 + b(t)y + c(t)$$

After fixing (at most) three solutions $x_0, x_1, x_2 \in \mathcal{S}(E)$, the "limit object" $\mathcal{S}(E)$ has the structure of the projective line $\mathbb{P}^1(\mathbb{C})$ together with all complex algebraic geometry living in

$$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \ldots$$



This means that for every $n \geq 5$, there are arbitrary large families of algebraic relations among the elements of $\mathcal{S}(E)$: there are families of closed algebraic subvarieties $(Z(\alpha) : \alpha \in S)$ of $\mathbb{A}^5$ such that

$$Z(\alpha) \cap S(E)^5 \text{ are pairwise distinct subsets of } S(E)^5.$$

Zilber dichotomy for order one equations

Consider $(E) : f(t, y, y') = 0$ an order one differential equation over $\mathbb{C}(t)$.

Theorem (Hrushvoski, 90's going back to Shelah 70's)

Up to a generically finite cover, there are only two possible models for the FOL structure $\mathcal{S}(E)$ of solutions of (E):

(1) *either the solutions of (E) are classical meromorphic functions (of the class $\mathcal{C}_0$).*
In that case, after fixing at most three solutions, the structure $\mathcal{S}(E)$ is a finite cover of the complex projective line $\mathbb{P}^1(\mathbb{C})$.

(2) *or the solutions of (E) are new meromorphic functions (live in $\mathcal{C}_1 \setminus \mathcal{C}_0$).*
In that case, the structure $\mathcal{S}(E)$ is a generically finite cover of the structure Δ_0 of a pure infinite set.

Question: What about algebraic differential equations of higher order?
The two previous models are still possible:

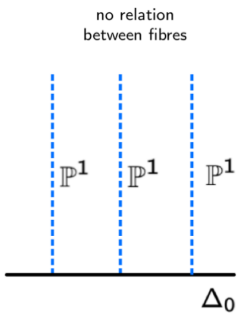
- the structure $\mathcal{S}(E)$ of solutions of the first Painlevé equation is the structure Δ_0 of a pure infinite set.
- The set of solutions of a linear differential equation of order 2 over $\mathbb{C}(t)$ has (after fixing a fundamental system of solutions) the structure of the complex affine plane $\mathbb{A}^2(\mathbb{C})$.

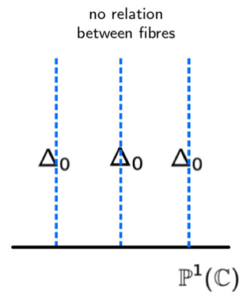
But there are more possibilities...

New phenomena in order two

- **Mixed (non-minimal) cases:**

$$(E_1):\begin{cases} y' = xy \\ x' = x^2(x-1)(x-\sqrt{2}) \end{cases} \quad (E_2):\begin{cases} y' = \frac{1}{x+y} \\ x' = x \end{cases} \quad (E_3):\begin{cases} y' = y^2(y-1)(y-\sqrt{2}) \\ x' = x \end{cases} \quad \dots$$





$\Delta_0 \times \mathbb{P}^1(\mathbb{C})$

- **Modular (non-disintegrated) examples** (Hrushovski): Consider the "Manin kernel"

$$(E): y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 + \left(\frac{1}{t-y} + \frac{1}{1-t} - \frac{1}{t}\right)y' + \frac{y(y-1)}{2t(t-1)(y-t)}$$

The limit object $\mathcal{S}(E)$ has the structure of a pure abelian group: there is an embedding of

$$\mathcal{S}(E) \to E(\mathcal{U}) \text{ where } E \text{ is the elliptic curve } y^2 = x(x-1)(x-t) \text{ over } \mathbb{C}(t)$$

such that for every closed complex subvariety $Z \subset E^n$,

$$Z \cap \mathcal{S}(E)^n = \left(\bigcup_{i=1}^{n}(a_i + H_i)\right) \cap \mathcal{S}(E)^n$$

where the H_i are subgroups and the a_i are torsion points of E^n.

Zilber trichotomy for minimal differential equations of order > 1

Let $(E) : F(t, y, y', \ldots, y^{(n)}) = 0$ be an order $n > 1$ algebraic differential equation over $\mathbb{C}(t)$.

Theorem (Hrushovski, 90's)

Assume that the generic solutions of (E) are new meromorphic functions.

(i) *Either the structure $\mathcal{S}(E)$ is disinstegrated (or trivial): for every "definable embedding" $\mathcal{S}(E) \to X(\mathcal{U})$, there are (at most) countably many closed*

$$\text{gen. finite to finite correspondences } \{R_i \subset X \times X, i \in I\} \text{ over } \mathbb{C}(t)$$

such that for every irreducible $\mathbb{C}(t)$-subvariety $Z \subset X^n$, there is a subset S_Z of ordered pairs of elements in $\{1, \ldots, n\}$and a function:

$$h_Z : S_Z \to I \text{ such that } Z \cap \mathcal{S}(E)^n_{gen} = \Big(\bigcap_{(i,j)\in S_Z} \pi^{-1}_{i,j}(R_{h_Z(i,j)})\Big) \cap \mathcal{S}(E)^n_{gen}$$

where $\pi_{i,j} : X^n \to X^2$ is the projection on the i^{th} and j^{th} coordinates.

(ii) *or the structure $\mathcal{S}(E)$ has the structure of a pure abelian group : there is a "definable embedding" of $\mathcal{S}(E)$ in $A(\mathcal{U})$ where A is a simple non isotrivial Abelian variety/$\mathbb{C}(t)$ and*

$$\text{for every closed } \mathbb{C}(t)\text{-subvariety } Z \text{ of } A^n, Z \cap \mathcal{S}(E)^n = \Big(\bigcup_{i=1}^{n}(a_i + H_i)\Big) \cap \mathcal{S}(E)^n$$

where the H_i are subgroups and the a_i are torsion points of A^n.

Moreover, if the equation (E) is autonomous , then only (i) occurs.

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Summary of the analysis so far

Consider any autonomous differential equation

$$(E) : F(y, y', y'', \dots, y^{(n)}) = 0 \text{ with } n > 1.$$

- The structure $\mathcal{S}(E)$ is minimal if and only if there are no combination of

$$(A1), (C1), (C2), (P_1), \dots (P_{n-1})$$

  producing a generic solution of (E).
- In that case, the structure of $\mathcal{S}(E)$ is automatically disintegrated by Hrushovski theorem. This can be summarized as: for every generic solutions $y_1, \dots, y_r$ of (E)

$$trdeg_{\mathbb{C}}(y_1, y_1' \dots, y_1^{(n-1)}, \dots, y_r, y_r' \dots, y_r^{(n-1)}) = n.r$$

  unless for some $i \neq j$,

$$trdeg_{\mathbb{C}}(y_i, y_i' \dots, y_i^{(n-1)}, \dots, y_j, y_j' \dots, y_j^{(n-1)}) = n.$$

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  producing a generic solution of (E).
- In that case, the structure of $\mathcal{S}(E)$ is automatically disintegrated by Hrushovski theorem. This can be summarized as: for every generic solutions $y_1, \dots, y_r$ of (E)

$$trdeg_{\mathbb{C}}(y_1, y_1' \dots, y_1^{(n-1)}, \dots, y_r, y_r' \dots, y_r^{(n-1)}) = n.r$$

  unless for some $i \neq j$,

$$trdeg_{\mathbb{C}}(y_i, y_i' \dots, y_i^{(n-1)}, \dots, y_j, y_j' \dots, y_j^{(n-1)}) = n.$$

- (mixed cases) On the other hand, if one can perform a semi-minimal analysis : a combination of

$$(A1), (C1), (C2), (P_1), \dots (P_{n-1})$$

  where each (P_i) is applied to a strongly minimal equation, one can produce a decomposition of $\mathcal{S}(E)$ into strongly minimal sets and describe as above the "structure" of the algebraic relations among solutions of $\mathcal{S}(E)$ as above.

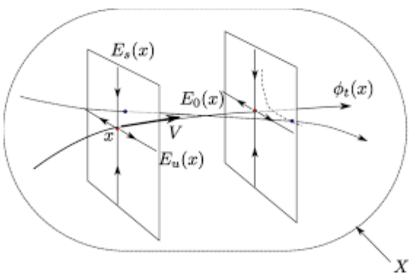
The main theorem

Theorem (J.)

Let $M \subset \mathbb{R}^n$ be a real-algebraic subset of the Euclidean subset. Assume that M is a compact surface, which is regular and that:

the restriction of the Euclidean metric to M has negative curvature at every point $p \in M$.

If $Geo(M)$ denotes the (autonomous) differential equation of unitary geodesics on M then the structure $\mathcal{S}(Geo(M))$ is minimal and disintegrated.



The first-order structure $\mathcal{S}(E)$ formed by the solutions of (E) in a "universal" diff. field.

- algebraic integrability
- algebraic independence of the solutions of (E)

How to compute the structure of $\mathcal{S}(X, v)$?

Let (X, v) be an autonomous differential equation and denote by ϕ_t the local (complex) analytic flow of the differential equation (X, v) that we can consider on an open neighborhood of every point $p \in X(\mathbb{C})$.

- We first consider the usual suspects:
  - the rational integrals of (X, v) i.e. the rational functions $f \in \mathbb{C}(X)$ such that

    $$f \circ \phi_t = f \text{ when both sides do make sense}$$

  - the closed invariant complex varieties $Z \subset X$ such that

    $$\phi_t(Z(\mathbb{C})) \subset Z(\mathbb{C}) \text{ at some/every point } a \in Z$$

Example: (Hadamard) the geodesic equation $Geo(M)$ where M is a compact surface with negative curvature does not admit any non-constant rational integral.

- It is not enough information to determine the structure of $S(X, v)$!

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Example: (Hadamard) the geodesic equation $Geo(M)$ where M is a compact surface with negative curvature does not admit any non-constant rational integral.

- It is not enough information to determine the structure of $S(X, v)$!
  To gather more information about $S(X, v)$, we study differential-algebraic structures on X admitting v (or the local analytic flow of v) as an infinitesimal symmetry.
  - k-forms ω on X satisfying

    $$\phi_t^* \omega = \omega.$$

  - foliations $\mathcal{F}$ on X satisfying

    $$\phi_t^* \mathcal{F} = \mathcal{F}.$$

  - possibly "higher-order" differential algebraic structures on X.

I will describe a method based on the study of algebraically integrable foliations $\mathcal{F}$ on X admitting the vector field v as an infinitesimal symmetry.

Invariant foliations and invariant webs

Let (X, v) be an autonomous differential equation.

Definition

We say that a foliation $\mathcal{F}$ on X is an invariant foliation of the differential equation (X, v) if it is preserved under the local analytic flow of the vector field v. Algebraically, this is expressed as

$$\mathcal{L}_v(\mathcal{F}) \subset \mathcal{F}$$

where $\mathcal{L}_v : \Theta_X \to \Theta_X$ denotes the Lie derivative of the vector field v.

- This dual nature of the notion of invariant foliation makes this notion very flexible: invariant foliations can be recovered from both:
  - a dynamical analysis of the differential equation under study.
  - a Galois-theoretic analysis of a partial connection $\mathcal{L}_v$ on T_X along the vector field v on X.
- We will need a slight generalization of the notion of invariant foliation: we also consider foliations on generically finite cover of X. If

$$\pi : X' \dashrightarrow (X, v) \text{ is a generically finite cover,}$$

the vector field v extends uniquely to a rational vector field v' on X' which is regular on the open set where π is étale. An invariant foliation $\mathcal{F}'$ on (X', v') will be called an invariant web of (X, v).

Question: Given the complete list $\mathcal{F}_1, \ldots, \mathcal{F}_r$ of invariant foliations and webs of (X, v), can one recover the structure $\mathcal{S}(X, v)$?

One more example

Consider a linear differential equation

$$(E) : Y' = AY \text{ where } A \in \mathbb{C}(t)$$

where the Galois group a simple linear algebraic group G.

- Such linear differential equations do exist by inverse Galois theory.
- We can realize this differential equation geometrically as the generic fibre of

$$\pi : (X, v) \to (\mathbb{A}^1 \setminus S, \frac{d}{dt})$$

  where S is the set of singularities of the coefficients of A, $X = \mathbb{A}^n \times (\mathbb{A}^1 \setminus S)$ and v is "linear in the fibres".
- Denote by $\mathcal{F}_\pi$ the algebraically integrable foliation of codimension one tangent to the fibres of π. The foliation $\mathcal{F}_\pi$ is invariant under v. Using Galois correspondence, there are no proper foliations

$$0 \subsetneq \mathcal{F} \subsetneq \mathcal{F}_\pi$$

  which are both algebraically integrable and invariant under the vector field v.

Conclusion: There are no algebraically integrable foliation properly contained in $\mathcal{F}_\pi$, yet the solution set of $\mathcal{S}(E)$ is far from being minimal: all the solutions of (E) live in the class $\mathcal{C}_0$.

- Similar examples can be realized for autonomous differential equations using a simple Abelian variety as the Galois group.

Computing $\mathcal{S}(X, v)$ using invariant foliations

Eliminating the previous examples, we get the following theorem:

Theorem (J.)

Let (X, v) be a autonomous differential equation of dimension $n > 1$. Assume that:

(1) *there are no algebraically integrable webs of foliationsinvariant under the vector field v.*

(2) *the generic solutions of (X, v) are algebraically independent from all classical meromorphic functions (i.e. in the class $\mathcal{C}_0$).*

Then the structure $\mathcal{S}(X, v)$ of the set of solutions of (X, v) is minimal and disintegrated.

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Then the structure $\mathcal{S}(X, v)$ of the set of solutions of (X, v) is minimal and disintegrated.

Let $M \subset \mathbb{R}^n$ be a compact regular real-algebraic surface of the Euclidean space with negative curvature. We will apply this theorem to a complexification (X, v) of the equation

$$\text{such that } X(\mathbb{R}) = SM \text{ and } \overline{X(\mathbb{R})} = X \text{ (so } X \text{ has dimension three).}$$

By compactness of the real locus $X(\mathbb{R}) = SM$, the vector field $v_{\mathbb{R}}$ induced by v can be integrated into a 1-parameter subgroup

$$\phi_t : SM \to SM, t \in \mathbb{R}$$

Theorem 2 $\Rightarrow$ Theorem 1: We show that the failure of (1) or (2) contradicts the dynamical properties of the flow $(\phi_t)_{t\in\mathbb{R}}$ described by Anosov by means of hyperbolic dynamics.

(2) can also be recovered from a dynamical analysis of $Geo(M)$

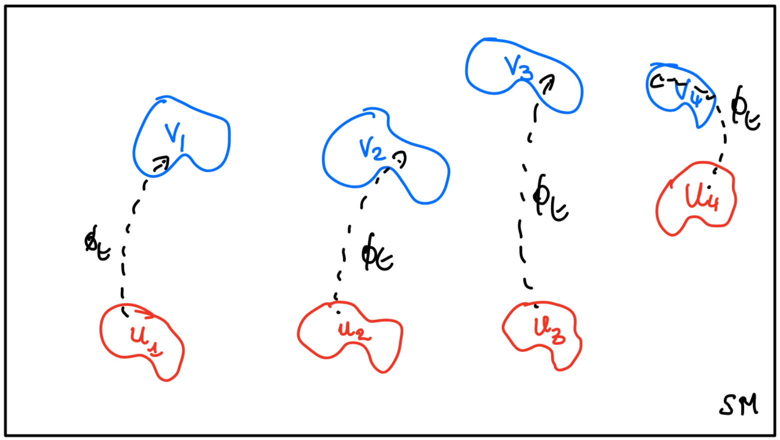
- Assume for the sake of a contradiction that (X, v) does not satisfy (1). Using standard arguments from stability theory, we show that this implies that for $n \gg 0$,

$$(X, v)^n \text{ admits a dominant rational integral } \rho : X^n \dashrightarrow \mathbb{P}^1.$$

After taking the real points, we obtain a real-analytic function (away from the singular locus)

$$\rho_{\mathbb{R}} : SM^n \dashrightarrow \mathbb{P}^1(\mathbb{R}) \text{ constant on the geodesics (when it makes sense).}$$

- Anosov showed that the geodesic flow of a compact surface is always mixing



Given $U_1, \ldots, U_n, V_1, \ldots, V_n$, there is a common time t such that

$$\phi_t(U_i) \cap V_i \neq \emptyset, i = 1, \ldots n.$$

- In the product SM^n, it means that ρ always takes a common value on $U_1 \times \ldots \times U_n$ and $V_1 \times \ldots \times V_n$. Since open sets of this form form a basis of the topology on SM^n, $\rho_{\mathbb{R}}$ hence ρ is constant contradiction.

Thank you for your attention!