A model-theoretic analysis of geodesic equations in negative curvature

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Virtual Event: Geometry and Dynamics of Foliations

- The subject of my talk is an approach to algebraic differential equations originating from the model-theoretic study of universal differential fields called differentially closed fields (Blum 68', Shelah 73', Rosenlicht 74') .
- The basic quantifier free formulas are systems (Σ) of algebraic differential equations and one studies the associated definable set: the set of solutions of (Σ) in one of these universal differential fields.

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- The basic quantifier free formulas are systems (Σ) of algebraic differential equations and one studies the associated definable set: the set of solutions of (Σ) in one of these universal differential fields.
- This approach is based on a powerful structure theory for definable sets in certain first-order theories based on a principle known as Zilber's trichotomy:

  there are essentially three kinds of "indecomposable" definable sets.
- The study of Zilber's trichotomy has been a driving force in model theory starting with the work of Hrushovski and Zilber on Zariski geometries in the 90's providing effective conditions for this principle to hold (differential algebra, difference algebra, Erdös geometry).

- *Locally modular case*: studied in the 90's by Hrushovski and Pillay in connection with Hrushovski's proof of Mordell-Lang conjecture for function fields of characteristic 0.

Example

Let $\pi : \mathcal{E} \to \mathbb{A}^1$ be a non-isotrivial family of elliptic curves and $\hat{\pi} : \hat{\mathcal{E}} \to \mathbb{A}^1$ the universal vector extension of $\mathcal{E}$.
There is a unique vector field v on $\hat{\mathcal{E}}$ extending the vector field $\frac{d}{dt}$ on $\mathbb{A}^1$ and compatible with the group structure.

In that case, $(\hat{\mathcal{E}}, v)$ is of dimension 3. $(\hat{\mathcal{E}}, v)$ does not admit a rational integral but there are countably many invariant algebraic curves.

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- *Non locally modular case*: algebraic differential equations with admits a differential Galois theory in the sense of Kolchin. $(\pm\epsilon)$

Example

Linear (possibly inhomogeneous) differential eaquations, Ricatti equations and elliptic differential equations

If a differential equation (X, v) does not admit a rational integral then it admits finitely many maximal invariant algebraic subvarieties.

- *Disintegrated case (with trivial forking geometry)* constituted of algebraic differential equations whose solutions are highly transcendental functions.

Example (Painlevé equations, Pillay-Nagloo '06 on the model theory side)

$$(E) : y'' = 6y^2 + t.$$

- The solutions of Painlevé equations define "new transcendental functions": no solution of (E) can be obtained by successive resolutions of
  - algebraic differential equations of order one $f(y,' y) = 0$.
  - linear differential equations, higher dimensional versions of elliptic equations.
- If $y_1, \ldots, y_n$ are distinct solutions of (E) then

$$(y_1, y_1', \ldots, y_n, y_n') \text{ are } \mathbb{C}(t)\text{-algebraically independent.}$$

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Theorem (Hrushovski-Sokolovic 96')

ZIlber's trichotomy holds for algebraic differential equations:

- *Any algebraic differential equation can be broken down into finitely many "indecomposable" pieces.*
- *Each of these indecomposable pieces belongs to exactly one of the three categories discussed above.*

Question

Is it possible to describe effectively such a decomposition for certain differential equations that appear in real life?

An idea of my PhD: the geodesic differential equation of a compact Riemannian surface with negative curvature is a perfect test problem:

- It is a non-completely integrable Hamiltionian system with two degrees of freedom, like many other interesting systems of classical mechanics.
- As shown by Anosov ('69), the dynamic of the vector field satisfies global hyperbolic properties. It makes it easier to study than other non-integrable Hamiltonian system which are closer to completely integrable ones.

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To apply Zilber's trichotomy, we need to work with algebraically presented Riemannian manifold and to complexify the differential equation.

Setting: We start with (X, g) a pseudo-Riemannian algebraic variety over $\mathbb{R}$ that is a smooth algebraic variety endowed with a non degenerate symmetric 2-form over $\mathbb{R}$ such that:

- $X(\mathbb{R})$ is Zariski-dense in X.
- $(X(\mathbb{R}), g_{\mathbb{R}})^{an}$ is a real-analytic Riemannian manifold (of dimension two) with negative curvature.

Geodesic differential equations

Let (M, g) be a compact Riemaniann manifold of dimension 2.

- The geodesic differential equations of (M, g) is the Hamiltonian system on TM associated with the "free" Hamiltonian:

$$H(x, y) = \frac{1}{2} g_x(y, y)$$

  We represent it as a pair (TM, v_H) where v_H is a vector field on TM.

- When the data is algebraic, the Hamiltonian defines a rational integral:

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- All the fibres of H have the same analysis with respect to Zilber's trichotomy (although the fibres with negative energies don't have real points).

Definition

The (unitary) geodesic differential equation of (M, g) is the differential equation of dimension three such that:

- The underlying manifold is the sphere bundle $SM \subset TM$ of M.
- It is given by the restriction to SM of the vector field v_H on TM.

We now assume that (M, g) has everywhere negative (but in general non constant) curvature.

- *Global hyperbolic structure:* there is a decomposition

$$T_{SM} = E^s \oplus E \oplus E^u$$

into $(d\phi_t)_{t\in\mathbb{R}}$-invariant continuous line bundles such that E is the direction of the vector field, $(d\phi_t)_{t\in\mathbb{R}}$ is exponentially contracting on E^s and exponentially expanding E^u.

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  into $(d\phi_t)_{t\in\mathbb{R}}$-invariant continuous line bundles such that E is the direction of the vector field, $(d\phi_t)_{t\in\mathbb{R}}$ is exponentially contracting on E^s and exponentially expanding E^u.
- *Periodic orbits*: Periodic points of $(\phi_t)_{t\in\mathbb{R}}$ are dense in SM.
- *Ergodic and mixing properties* The dynamic of $(SM, (\phi_t))_{t\in\mathbb{R}}$ is topologically (weakly) mixing that is:

$$(SM, (\phi_t)_{t\in\mathbb{R}}) \text{ and } (SM \times SM, (\phi_t \times \phi_t)_{t\in\mathbb{R}}) \text{both admit a dense orbit.}$$

  This implies that for all $n \geq 3$, $(SM^n, (\phi_t \times \ldots \times \phi_t)_{t\in\mathbb{R}})$ admits a dense orbit too.
  Moreover, the leaves of the $\mathcal{C}^1$-foliations W^{ss} and W^{su} tangent to E^s and E^u respectively are dense in M (Plante).

Theorem (Qualitative description of the structure in negative curvature)

Let (X, g) be a pseudo-Riemannian variety over $\mathbb{R}$ such that the real-analytification $(X(\mathbb{R}), g_{\mathbb{R}})^{an}$ is a compact Riemannian surface with negative curvature. The geodesic differential equation (SX, v) of (X, g) is:

(1) ***"indecomposable" or minimal**: the generic solution of (SX, v) does not lie in a differential field of the form*

$$(\mathbb{C}(t), \frac{d}{dt}) = (K_0, \delta_0) \subset (K_1, \delta_1) \subset \ldots \subset (K_n, \delta_n).$$

where each elementary step $(K_i, \delta_i) \subset (K_{i+1}, \delta_{i+1})$ is either:

- (i) *an algebraic extension or a pure extension by constants.*
- (ii) *a strongly normal extension in the sense of Kolchin.*
- (iii) *an extension generated by solutions of differential equations order < 3.*

(2) ***disintegrated**: the solutions share "few algebraic relations": There are two subsets $\mathcal{Z}_1 \subset SX$ and $\mathcal{Z}_2 \subset SX \times SX$ which are (at most) countable union of proper closed invariant algebraic varieties such that:*

any collection $(\gamma_i, i \in \mathcal{I})$ of solutions of (SX, v) form a $\mathbb{C}$- algebraically independent set

unless $\gamma_i(0) \in \mathcal{Z}_1$ for some $i \in \mathcal{I}$ or $(\gamma_i(0), \gamma_j(0)) \in \mathcal{Z}_2$ for some $i \neq j \in \mathcal{I}$.

- **Qualitative/Quantitative**: To obtain a quantitative description, one needs to describe explicitly $\mathcal{Z}_1 \subset SX$ and $\mathcal{Z}_2 \subset SX \times SX$.
  - One can always choose $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) to be the union of all proper closed invariant subvarieties of (SX, v) (resp. of $(SX \times SX, v \times v)$).
  - In our case, these sets have at most countably many maximal elements. An interesting question is to study if there are only finitely many.
  - The exact description of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ might depend or not of the chosen compact Riemannian manifold and on the complexification.

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- **Generic solution**: An analytic solution of (SX, v) (seen as analytic curve $\gamma : \mathbb{D} \to SX$) is generic if its image is not contained (and equivalently does not meet) $\mathcal{Z}_1$.
  Since $\mathcal{Z}_1$ is an F_σ-set of $X(\mathbb{R})^{an}$ of measure zero, in various senses, (1) applies to "most" geodesics.

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  Since $\mathcal{Z}_1$ is an F_σ-set of $X(\mathbb{R})^{an}$ of measure zero, in various senses, (1) applies to "most" geodesics.
- **Key idea**: Study algebraic correspondences between (SX, v) and "indecomposable" differential equations in each of the classes of Zilber's trichotomy.
  The study of each of these three classes has a different flavor based on the structural model-theoretic properties underlying Zilber's trichotomy.

Orthogonality to the constants

Let (X, v) and (Y, w) be two differential equations. We endow $X \times Y$ with the product vector field $v \times w$.

- A non-trivial algebraic correspondence between (X, v) and (Y, w) is a proper closed invariant algebraic subvariety Z of $X \times Y$ projecting generically on both factors.
- We say that a differential equation (X, v) is orthogonal to the constants if
  - (X, v) does not admit a rational integral.
  - There are no non-trivial algebraic correspondences between (X, v) and a differential equation admitting a Galois theory in the sense of Kolchin.

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Theorem (Corps differentiels et Flots géodésiques I)

Let (X, v) be an absolutely irreducible differential equation over $\mathbb{R}$. Assume that there exists a compact $K \subset X(\mathbb{R})^{an}$ Zariski-dense in X, constituted of regular points of $X(\mathbb{R})^{an}$ and invariant under (the local real analytic flow of) the vector field v.
Denote by $(K, (\phi_t)_{t\in\mathbb{R}})$ the complete flow induced by v on K. If $(K, (\phi_t)_{t\in\mathbb{R}})$ is topologically weakly mixing then (X, v) is orthogonal to the constants.

In particular, this is an obstruction for completely solvability by linear differential equations, elliptic differential equations and their higher-dimensional versions.

Theorem

Let p be a prime number. Assume that (X, v) is a differential equation of dimension p satisfying:

(i) *(X, v) is orthogonal to the constants.*

(ii) *Every foliation $\mathcal{F}$ on X of rank $r \in \{1, \ldots, p-1\}$ which is invariant under the vector field v has a Zariski-dense leaf.*

(iii) *Every p-web $\mathcal{W}$ of foliations by curves on X which is invariant under the vector field v has a Zariski-dense leaf.*

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Then the conclusion of the main theorem holds.

- In (ii), a foliation $\mathcal{F}$ of rank r on X is a saturated coherent subsheaf of rank r of the locally-free sheaf $\Theta_{X/k}$ stable under Lie bracket. It is called invariant under v if
$$[v, \mathcal{F}] \subset \mathcal{F}$$
- In (iii), a (generically smooth) r-web W of foliations by curves on X is a closed subvariety $W \subset \mathbb{P}(T_X)$ such that all irreducible components of W dominate X and $\pi_{|W} : W \to X$ is generically finite.
The vector field v has a first projective prolongation $\mathbb{P}(v)$ to $\mathbb{P}(T_X)$ from which derives the notion of invariance for webs.

A final picture

- On a compact surface with negative curvature, the geodesic equation is "indecomposable" and disintegrated.
- On an ellipsoid, the geodesic equation is decomposable and all its "indecomposable pieces" admit a Galois theory in the sense of Kolchin.
- For surfaces of revolution, the geodesic equation is always decomposable. In some cases, one observes a mixed situation where some of the "indecomposable" pieces admit a Galois theory in the sense of Kolchin and others are disintegrated.

Thank you for your attention!