LECTURE 4. ELIMINATION OF IMAGINARIES

Definition 4.1. A complete theory T admits the *elimination of imaginaries* if for every definable equivalence relation E on some definable set $D \subset M^n$ of some model $M \models T$, there exists a definable function $f : D \to M^s$ such that

$$xEy \Leftrightarrow f(x) = f(y).$$

One can then identify D/E with the definable set $f(D)$ and take quotient within the category of definable sets. The goal of this lecture is to prove the following theorem:

Theorem 4.2 (Poizat, 1983)**.** *The theory* DCF_0 *eliminates imaginaries.*

There is an abstract procedure in model theory which from a theory T produces a theory T^{eq} which is bi-interpretable with T and eliminates imaginaries. Understanding this process explicitly is an important source of applications of model theory to algebra. The first example of theory which eliminates imaginaries is the theory ACF_0 of algebraically closed fields of characteristic zero.

Exercise 4.3. *Show that the theory of the infinite set with equality does not eliminate imaginaries.*

4.1. **Description of the types.** Let T be a complete theory and R a substructure of some model K of T (e.g. $K \models \mathrm{DCF}_0$ and R is a differential subring).

Definition 4.4. A partial type over R with variables $x = (x_1, \ldots, x_n)$ is a collection of formulas $\pi(x) = \{\psi(x) \mid \psi(x)\ \mathcal{L}(R) - \text{formula}\}$ such that for any finite subset S of $\pi(x)$

$$K \models \exists x \bigwedge_{\psi(x) \in S} \psi(x)$$

A *complete type* is a partial which is maximal for the partial order defined by inclusion.

Remark 4.5. A partial type $\pi(x)$ is complete if and only if for every formula $\phi(x)$, either $\phi(x) \in \pi(x)$ or $\neg\phi(x) \in \pi(x)$.

Proof. Let $\pi(x)$ be a partial type and $\phi(x)$ a formula then $\pi(x) \cup \phi(x)$ or $\pi(x) \cup \neg\phi(x)$ is a partial type. Indeed, otherwise, there are finite subsets S_+ and S_- such that

$$K \models \neg(\exists x \bigwedge_{\psi(x) \in S_+} \psi(x) \wedge \phi(x) \text{ and } K \models \neg(\exists x \bigwedge_{\psi(x) \in S_-} \psi(x) \wedge \neg\phi(x)$$

It follows that $K \models \forall x (\bigwedge_{\psi(x) \in S_+ \cup S_-} \psi(x)) \to (\phi(x) \wedge \neg\phi(x))$ which contradicts that $\pi(x)$ is a partial type. $\square$

Lemma 4.6. *Assume that* T *eliminates quantifiers. Then the notions of partial and complete types do not depend on the chosen model* K *extending* R.

Proof. Assume L is another model extending R, $\pi(x)$ is a partial type in the model L and S a finite subset. By QE, there exists a quantifier-free sentence ϕ with parameters from R such that

$$T \vdash \exists x \big(\bigwedge_{\psi(x) \in S} \psi(x) \big) \leftrightarrow \phi$$

Since ϕ is quantifier-free, it is a boolean combination of sentences of the form

$$Q(t_1(r_1, \ldots, r_s), \ldots, t_k(r_1, \ldots, r_s))$$

where Q is a k-ary relation symbol in the language, $t_1, \ldots, t_k$ are terms and $r_1, \ldots, r_s \in R$. By definition of substructures, we have that

$$L \models \phi \text{ iff } (t_1^R(r), \ldots, t_k^R(r)) \in Q^L \cap R^k = Q^R \text{ iff } K \models \phi$$

as required. □

Example 4.7. Let $a = (a_1, \ldots, a_n) \in K^n$. Then

$$\mathrm{tp}(a/R) = \{\phi(x) \mid K \models \phi(a)\}$$

is a complete type with n variables over R. The types of this form are said to be *realized in* K.

Exercise 4.8. *Every complete type is realized in some elementary extension L of K.*

For now on, if R is a differential ring, we denote by $S_n(R)$ the space of types with n variables and parameters in R in the sense of the theory DCF_0.

Corollary 4.9 (Description of types)**.** *Assume that k is a differential field. The correspondence*

$$p \mapsto I = \{f \in k\{X_1, \ldots, X_n\} \mid \text{“}f(x) = 0\text{''} \in p\}$$

is a bijection between $S_n(k)$ and the set of prime differential ideals *of $k\{X_1, \ldots, X_n\}$.*

Proof. We first show that I is a prime ideal. Take $a \models p$. By enlarging K if necessary, we can find a realization a of p in a model L of DCF_0. By construction of a, we have that

$$I = I(a/k) = \{f \in k\{X_1, \ldots, X_n\} \mid f(a) = 0\}.$$

This means that I is the kernel of the evaluation map at a so that I is a prime differential ideal. Now since every formula is equivalent to boolean combination of formulas of the form $f(x) = 0$, a type $p \in S_n(k)$ is determined by the function

$$f(x) \mapsto \chi_p : \begin{cases} 0 \text{ if “}f(x) = 0\text{''} \in p \\ 1 \text{ otherwise} \end{cases}$$

which is the characteristic function of the subset I in $k\{X_1, \ldots, X_n\}$. Surjectivity follows from the differential Nullstellensatz. Indeed, if $g_1, \ldots, g_n \notin I$ then $J = \{I, g_1 \cdots g_n\}$ is a radical differential ideal distinct from I so that $V(J) \subsetneq V(I)$. It follows that the collection of formulas

$$f(x) = 0 \text{ for } f \in I \text{ and } g(x) \neq 0 \text{ for } g \notin I$$

is finitely consistent. □

Corollary 4.10 (Definable and algebraic closure). *Let A be a subset of K. Then*

$$\mathrm{dcl}(A) = \mathbb{Q}\langle A\rangle \text{ and } \mathrm{acl}(A) = \mathbb{Q}\langle A\rangle^{alg}$$

where $\mathbb{Q}\langle A\rangle$ is the differential field generated by A, dcl *is the model-theoretic definable closure and* acl *denotes the model-theoretic algebraic closure.*

Proof. Clearly, $\mathbb{Q}\langle A\rangle \subset \mathrm{dcl}(A)$ and $\mathbb{Q}\langle A\rangle^{alg} \subset \mathrm{acl}(A)$ and we prove the reverse inclusions. Set $k = \mathbb{Q}\langle A\rangle$ and consider $x \in \mathrm{acl}(A)$, $I(x/k) \subset k\{X\}$ the associated ideal and f the minimal polynomial in $I(x/k)$ with respect to $\ll$. To show that $x \in k^{alg}$, we need to show that $d = \mathrm{ord}(f) = 0$.

Assume otherwise. We claim that by induction on n, the conditions

$$x_1 \neq \cdots \neq x_n, f(x_i) = 0 \text{ and } g(x_i) \neq 0 \text{ for all g with } ord(g) < d\}$$

are consistent (exercise). Since $I(x_i/k) = I(x/k)$ for all i, this contradicts that $x_i \in \mathrm{acl}(k)$. □

4.2. **Saturated models.** Instead of working in arbitrary models of the theory DCF_0, it is convenient to work in saturated models.

Lemma 4.11. *The theory DCF_0 is ω-stable, that is, for any differential ring R, we have*

$$|S_n(R)| = |R|.$$

Proof. A theory T in a countable language if for any infinite set of parameters A, we have that $|S_1(A)| \leq |A|$. Denote by k the differential field generated by A. Since A is infinite, we have $| k |=| A |$ and the restriction morphism

$$S_1(k) \to S_1(A)$$

is a bijection. Using the previous corollary together with the first theorem of Ritt (Lecture 2), we obtain that $|S_1(A)| = |S_1(k)| \leq |k| = |A|$ as required. □

Fact 4.12. *Let κ be a regular cardinal. There exists a model $\mathcal{U}$ of DCF_0 of cardinal κ which is saturated in the sense that*

> *for every differential subfield k of size less than κ, every type $p \in S(k)$ is realized in $\mathcal{U}$*

and this model is unique *up to non unique isomorphism. A saturated model has the following properties*

(i) universality*: every differential field of size less than κ embeds in $\mathcal{U}$*
(ii) homogeneity*: any two tuples of elements which realize the same type over some differential field k of size less than κ are conjugated by automorphism $\sigma \in \mathrm{Aut}_\partial(\mathcal{U}/k)$ of $\mathcal{U}$ fixing k.*

We refer to [MMP96, Section 4.3] for a proof of this fact valid for arbitrary ω-stable theory. The existence part of the proof is similar to the proof is similar to the proof of existence of a differentially closed fields but requires some nontrivial cardinal arithmetic.

Model-theoretic convention. Fix κ a regular cardinal which is greater than all the differential fields we are interested in (e.g. take $\kappa > |\mathbb{C}| = |\mathcal{M}(U)|$ for every open domain U inside of $\mathbb{C}$) and fix $\mathcal{U}$ a saturated model of DCF_0. Unless otherwise stated, by a differential field, we always mean a differential subfield of $\mathcal{U}$.

Exercise 4.13. *Show that $b \in \mathrm{dcl}(A)$ if and only if b is fixed by every automorphism of $\mathcal{U}$ fixing A pointwise. Show that $b \in \mathrm{acl}(A)$ if and only if b has a finite orbit under the group of automorphisms of $\mathcal{U}$ fixing A pointwise.*

4.3. **Canonical parameters.** *Fix once for all $K \models \mathrm{DCF}_0$ a saturated model.*

Definition 4.14. Let $\phi(x,a)$ be a formula. We say that $\phi(x,a)$ is defined over a differential subfield $k \subset K$ such that there exists a formula $\psi(x,b)$ with parameters $b = b_1, \dots, b_n$ from k such that

$$K \models \forall x \Big(\psi(x,b) \leftrightarrow \phi(x,a) \Big).$$

Lemma 4.15. *Let $\phi(x,a)$ be a formula and let k be a differential subfield of K. Then $\phi(x,a)$ is defined over k* if and only if *for every $\sigma \in \mathrm{Aut}_\partial(K)$,*

$$\sigma \text{ fixes } k \text{ pointwise } \Rightarrow \sigma(D) = D \text{ setwise}$$

where $D = \phi(K,a)$ is the definable set defined by $\phi(x,a)$.

Proof. The direct implication is obvious. To prove the converse, consider a differential subfield k such that for every $\sigma \in \mathrm{Aut}_\partial(K)$, if σ fixes k pointwise then $\sigma(D) = D$ setwise. Fix $b \in D$. By homogeneity of K, the assumption implies that any other realization of $p = \mathrm{tp}(b/k)$ also lies in D. So that

$$\mathrm{DCF}_0 \vdash p(x) \to \phi(x,a)$$

By compactness, we can find a formula $\psi_b(x) \in p(x)$ with parameters from k, such that $\mathrm{DCF}_0 \vdash \psi_b(x) \to \phi(x,a)$ and $K \models \psi_b(b)$. Since this is true for any $b \in D$, we conclude that

$$\mathrm{DCF}_0 \vdash \phi(x,a) \leftrightarrow \bigvee_{b \in D} \psi_b(x)$$

Using compactness again, we obtain that $\mathrm{DCF}_0 \vdash \phi(x,a) \leftrightarrow \bigvee_{i=1}^n \psi_{b_i}(x)$ which shows that k is a field of definition of $\phi(x,a)$. □

This motivates the following definition:

Definition 4.16. Let $\phi(x,a)$ be a formula. We say that a tuple α from K is a *canonical parameter* for $\phi(x,a)$ if whenever σ is an automorphism of K then σ fixes c pointwise if and only if σ fixes the definable set $D = \phi(K,a)$ setwise.

Exercise 4.17. *Show that if α is a canonical parameter for $\phi(x,a)$ then $\phi(x,a)$ is defined over $\mathbb{Q}\langle\alpha\rangle$.*

Proposition 4.18. *The following properties are equivalent:*

(i) *$T = \mathrm{DCF}_0$ admits the elimination of imaginaries,*
(ii) *every formula admits a canonical parameter,*
(iii) *every Kolchin-closed set admits a canonical parameter.*

Proof. (i) $\Rightarrow$ (ii). Let $\phi(x,a)$ be a formula with $a = a_1, \dots, a_n$. Consider the definable equivalence relation $E(y,z)$ on K^n defined by

$$E(y,z) \text{ iff } K \models \forall x (\phi(x,y) \leftrightarrow \phi(x,z))$$

and denote by $f_E : K^n \to K^m$ the function witnessing elimination of imaginaries. We claim that $\alpha = f_E(a)$ is a canonical parameter of $\phi(x,a)$. Indeed, by construction

$$\sigma(\alpha) = \alpha \text{ iff } K \models \phi(x,a) \leftrightarrow \phi(x,\sigma(a)) \text{ iff } \sigma(D) = D.$$

(iii) $\Rightarrow$ (i) Let $E(y,z)$ be a definable equivalence relation on some definable set D defined over k. For $a \in D$, denote by

$$[a]_E = \{x \in D \mid xEa\}$$

and by $\overline{[a]_E}$ its Kolchin-closure. We first claim that

Claim. *$\overline{[a]_E} = \overline{[b]_E}$ iff aEb.*

Proof of the claim. Clearly $aEb \Rightarrow [a]_E = [b]_E \Rightarrow \overline{[a]_E} = \overline{[b]_E}$. Conversely, if $\overline{[a]_E} = \overline{[b]_E}$ then $[a]_E$ contains a dense open Kolchin-subset U_a of $\overline{[a]_E}$ and so does $[b]_E$. Since any two dense Kolchin subset intersect, we have $U_a \cap U_b \neq \emptyset$ which implies (by transitivity) that aEb as required. ☐

Now fix $a \in D$, $p = \mathrm{tp}(a/k)$ and fix:

(i) α for a canonical parameter for $\overline{[a]_E}$,
(ii) $\psi(x,y)$ a formula such that $\overline{[a]_E} = \psi(K,a)$.

Note that since $[a]_E$ is $k\langle a\rangle$-definable so is $\overline{[a]_E}$ and therefore $\alpha \in k\langle a\rangle$ and there exists a k-definable function $f_a : D \to K^m$ such that $f_a(a) = \alpha$.

Claim. *If $b_1, b_2 \models p$ then b_1Eb_2 iff $f_a(b_1) = f_a(b_2)$.*

Proof of the claim. Using an automorphism argument, if $b \models p$ and $\beta = f_a(b)$ then β satisfies the analogue of (i) and (ii) for $\overline{[b]_E}$.

Hence if $\beta_i = f_a(b_i)$ and $\beta_1 = \beta_2$ then

$$\overline{[b_1]_E} = \psi(K,b_1) = \psi(K,b_2) = \overline{[b_2]_E}$$

and b_1Eb_2 by the previous claim. Conversely, assume that b_1Eb_2 so that $\overline{[b_1]_E} = \overline{[b_2]_E}$. Consider $\sigma \in \mathrm{Aut}_\delta(K/k)$ such that $\sigma(b_1) = b_2$ then by definition

$$\sigma(\overline{[b_1]_E}) = \overline{[b_2]_E} = \overline{[b_1]_E}$$

and it follows using that β_1 is fixed by every automorphism fixing $\overline{[b_1]_E}$ that

$$\beta_1 = \sigma(\beta_1) = \sigma(f_a(b_1)) = f_a(\sigma(b_1)) = f_a(b_2) = \beta_2$$

as required. ☐

Since this is true for any point $a \in D$, we can find by compactness a decomposition

$$D = D_1 \cup \ldots \cup D_r \text{ and } k\text{-definable functions } f_i : D_i \to K^{n_i}$$

such that for every $b, c \in D_i$, bEc iff $f_i(b) = f_i(c)$. We conclude the proof by building by induction on $i \leq r$ a k-definable function $g_i : D_1 \cup \ldots \cup D_i \to K^{m_i}$ with the same property: assume that g_i has been already build for $i < r$ and consider S_{i+1} the k-definable subset of D_{i+1} given by

$$S_{i+1} = \{x \in D_{i+1} \mid \exists z \in D_1 \cup \ldots \cup D_i \text{ such that } zEx\}$$

and consider $G \subset S_{i+1} \times K^{m_i}$ defined by

$$G := \{(x,y) \in S_{i+1} \times K^{m_i} \mid \exists z \in D_1 \cup \ldots \cup D_i \mid zEx \text{ and } g_i(z) = y\}$$

By the induction hypothesis, G is the graph of a k-definable function g. The function

$$g_{i+1}(x) = \begin{cases} g_i(x) \text{ if } x \in D_1 \cup \ldots \cup D_i \\ g(x) \text{ if } x \in S_{i+1} \\ f_{k+1}(x) \text{ otherwise.} \end{cases}$$

is an extension of g_i satisfying the required properties. This concludes the proof of the proposition. □

4.4. **Field of definition of an ideal.**

Theorem 4.19 (André Weil)**.** *Every ideal I of $K[X_1, \ldots, X_n]$ admits a smallest field of definition, that is, there is a smallest subfield k of K with the property that I is generated by polynomials with coefficients in k. Furthermore, k is fixed pointwise by every automorphism of K fixing I setwise.*

Proof. Denote by M a basis of monomials of $K[\overline{X}]/I$ as a K-vector space. Each monomial u of $K[\overline{X}]$ can be uniquely written as

$$u = \sum_{m \in M} a_{u,m} m + f_u$$

where $f_u \in I, a_{u,m} \in K$.

Claim. *The field*

$$k = \mathbb{Q}[a_{u,m} \mid u \textit{ monomial of } K[\overline{X}], m \in M]$$

is the smallest field of definition of I.

- Step 1. *We show that k is a field of definition of I.*

For $f \in I$, we can write

$$f = \sum_{u \text{ mon. of } K[\overline{X}]} b_u u = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m \Big) + \sum_{m \in M} \Big(\sum_{u \text{ mon. of } K[\overline{X}]} b_u a_{u,m} \Big) \cdot m$$

Since by definition the left term lies in I and M is a K-basis of $K[\overline{X}]/I$, we conclude that all the coefficients of the right term must be zero and hence that

$$f = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m \Big).$$

It follows that I is generated by the $u - \sum_{m \in M} a_{u,m} m \in k[\overline{X}]$ where u ranges over all monomials of $K[\overline{X}]$ so that k is indeed a field of definition for I.

- Step 2. *Consider l another field of definition of I. We show that $k \subset l$.*

Note that every automorphism of K extends to an automorphism of $K[X_1, \ldots, X_n]$ by setting:

$$\sigma(\sum_{m \in mon. k[X]} f_m \cdot m) = \sum_{m \in mon. k[X]} \sigma(f_m) \cdot m$$

Since l is a field of definition of I, for every $\sigma \in \mathrm{Aut}(K/l)$, we have $\sigma(I) = I$. It follows that for every monomial u, we have

$$u = \sigma(u) = \sum_{m \in M} \sigma(a_{u,m}) \cdot m + \sigma(f_u)$$

By uniqueness of the decomposition, it follows that $\sigma(a_{u,m}) = a_{u,m}$ for every $\sigma \in \mathrm{Aut}(K/l)$ and every u, m. We have therefore shown that k is a subset of l. □

Corollary 4.20. *Every radical differential ideal I of $K\{X_1, \ldots, X_n\}$ admits a smallest differential field of definition. Furthermore, k is fixed pointwise by every differential automorphism of K fixing I setwise.*

Proof. By the Ritt-Raudenbush Theorem, we can find a finite set of differential polynomial such that

$$I = \{f_1, \ldots, f_n\}$$

Consider N large enough so that $f_1, \ldots, f_n \in K[X, X', \ldots, X^{(N)}]$ and set J for the ideal they generate. By Theorem 4.19, J has a smallest field of definition $k \subset K$. The differential field generated by k is the smallest differential field of definition of I. □

Exercise 4.21. *Using the differential Nullstellensatz, the previous corollary and Proposition 4.18, show that* DCF_0 *admits the elimination of imaginaries.*

Example 4.22. Denote by $\mathcal{C}$ the field of constants of K. Consider the action of an algebraic group G on an algebraic variety X and the equivalence relation on $X(K)$ given by:

$$xEy \text{ if and only if } x \text{ and } y \text{ lie in the same } G(\mathcal{C})\text{-orbit.}$$

There is a definable differential-algebraic function

$$h = X(K) \to K^n$$

which realizes the quotient $X(K)/G(\mathcal{C}) = X/E$. Taking $G = X = \mathrm{GL}_n$ acting on itself on the left, we can take

$$h : P \mapsto P' \cdot P^{-1}$$

Indeed, setting $A = P' \cdot P^{-1}$, the solutions of $h(Y) = A$ are the fundamental systems of solutions of the linear differential equations $Y' = AY$. Hence any two of them differ by multiplication by a constant matrix in $\mathrm{GL}_n(\mathcal{C})$. Another more subtle example is obtained by taking $G = \mathrm{PSL}_2$ and $X = \mathbb{P}^1$. In that case, we can take

$$h : y \mapsto (y''/y')' - 1/2(y''/y')^2.$$

This is called the Schwarzian derivative. The Schwarzian derivative is used in differential geometry to study projective structures on curves.

4.5. **References.** The elimination of imaginaries for differentially closed fields was defined and proved by Poizat in [Poi83]. We follow here the presentation from [Mar00].

References

[Mar00] David Marker. Model theory of differential fields. In *Model theory, algebra, and geometry*, volume 39 of *Math. Sci. Res. Inst. Publ.*, pages 53–63. Cambridge Univ. Press, Cambridge, 2000.

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