LECTURE 5. DIFFERENTIAL FORMS

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5.1. **Definition.** Let K/k be a field extension of characteristic zero and let M be a K-vector space. A *k-derivation on K with values in M* is an additive morphism

$$d_M : (K, +) \to (M, +)$$

satisfying

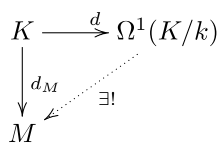
- (the Leibniz rule) $d_M(x \cdot y) = d_M(x) \cdot y + x \cdot d_M(y)$ for all $x, y \in K$
- (k-linearity) $d_M(x) = 0$ for all $x \in k$.

Notice that using the Leibniz rule, the second property is indeed equivalent to k-linearity of d_M when K are M are both equipped with their natural structure of k-vector space.

Lemma 5.1. *Let K/k be a field extension. There exists a (unique) pair $(\Omega^1(K/k), d)$ satisfying*

$(*)$ *$\Omega^1(K/k)$ is a K-vector space and d is a k-derivation on K with values in $\Omega^1(K/k)$*

satisfying the following universal property: for every pair (M, d_M) satisfying $(*)$, there exists a unique morphism of K-vector spaces $\Omega^1(K/k) \to M$ such that the following diagram commutes:



The pair $(\Omega^1(K/k), d)$ is called the module of *one-forms on K/k* or *the module of k-differentials on K*.

Proof. Consider $E = \mathrm{Span}_K\{\delta x \mid x \in K\}$ the K-vector space whose basis is given by symbols of the form δx for $x \in K$ and $\mathcal{R}$ the sub-vector space of E generated by all the elements of E of the form

$$\delta(x + y) - \delta x - \delta y, \delta(xy) - x\delta y - y\delta x \text{ and } \delta c$$

where x, y ranges over all the elements of K and c ranges over all the elements of k. We define

$$\Omega^1(K/k) = E/\mathcal{R} \text{ and } d : x \in K \mapsto x = \overline{\delta x} \in \Omega^1(K/k).$$

Certainly, $(\Omega^1(K/k), d)$ satisfies $(*)$. Now consider (M, d_M) another pair satisfying $(*)$. The function $\delta x \mapsto d_M x$ extends uniquely to a morphism of K-vector spaces $\phi : E \to M$. Since M satisfies $(*)$, we have $\phi(\mathcal{R}) \subset \mathrm{Ker}(\phi)$ so that ϕ factors through

$$\overline{\phi} : E/\mathcal{R} = \Omega^1(K/k) \to M.$$

By construction, for every $x \in K$,

$$\overline{\phi}(\delta x) = \phi(\delta x) = d_M x$$

which shows that the diagram in the lemma commutes. Uniqueness follows from the fact that $\Omega^1(K/k)$ is generated by the dx with $x \in K$ (exercise). $\square$

The Leibniz rule together with additivity are sufficient properties to ensure that the differential calculus happens "as intended". For example,

Exercise 5.2. *Let K/k be an extension of differential fields, $P(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ and $z_1, \ldots, z_n \in K$. Then*

$$d(P(z_1, \ldots, z_n)) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(z_1, \ldots, z_n) \cdot dz_i \tag{1}$$

Lemma 5.3. *Let K/k be an extension of differential fields and denote by ∂ the derivation on K. There is a unique* additive map $\mathcal{L}_\partial : \Omega^1(K/k) \to \Omega^1(K/k)$ *satisfying:*

- the Leibniz rule*: for all $a \in K$ and all $\omega \in \Omega^1(K/k)$*
$$\mathcal{L}_\partial(a \cdot \omega) = \partial(a) \cdot \omega + a \cdot \mathcal{L}_\partial(\omega).$$
- the chain rule*: $d \circ \partial = \mathcal{L}_\partial \circ d$.*

Proof. The uniqueness part is clear since the one-forms of the form dx and $x \in K$ generate $\Omega^1(K/k)$. Indeed, if $\omega = \sum_{i=1}^n \lambda_i \cdot dx_i$ then

$$\mathcal{L}_\partial(\omega) = \sum_{i=1}^n \mathcal{L}_\partial(\lambda_i \cdot dx_i) = \sum_{i=1}^n \Big(\partial(\lambda_i) \cdot dx_i + \lambda_i \cdot d(\partial(x_i))\Big).$$

To show existence, recall the construction of $\Omega^1(K/k) = E/\mathcal{R}$ from Lemma 5.1 and consider the map defined by the previous formula

$$L_\partial : \sum_{i=1}^n \lambda_i \cdot \delta x_i \mapsto \sum_{i=1}^n \Big(\partial(\lambda_i) \cdot \delta x_i + \lambda_i \cdot \delta(\partial(x_i))\Big)$$

which is a additive map from E to E. To show that it descends to $\Omega^1(K/k)$, we need to show that $L_\delta(\mathcal{R}) \subset \mathcal{R}$. Consider $x, y \in K$

$$\begin{aligned} L_\partial(\delta(xy) - x\delta y - y\delta x) &= \delta(\partial(xy)) - \partial(x)\delta(y) - y\delta(\partial(x)) - \partial(y)\delta(x) - x\delta(\partial(y)) \\ &= \Big(\delta(x\partial(y) + y\partial(x)) - \delta(x\partial(y)) - \delta(y\partial(x))\Big) \\ &+ \Big(\delta(x\partial(y)) - x\delta(\partial(y)) - \partial(x)\delta(y)\Big) \\ &+ \Big(\delta(y\partial(x)) - y\delta(\partial(x)) - \partial(y)\delta(x)\Big) \in \mathcal{R} \end{aligned}$$

as it is the sum of three elements from $\mathcal{R}$. The other generators of $\mathcal{R}$ satisfy (easier) identities (exercise). It follows that L_∂ factors through an additive map $\mathcal{L}_\partial : E/\mathcal{R} \to E/\mathcal{R}$ satisfying the required properties. ☐

Definition 5.4. Let K/k be an extension of differential fields. The additive map

$$\mathcal{L}_\partial : \Omega^1(K/k) \to \Omega^1(K/k)$$

given by Lemma 5.3 is called the *Lie-derivative of the derivation ∂.*

5.2. **Linear independence of one-forms.**

Theorem 5.5. *Let K/k be an extension of* fields *of characteristic zero and $(z_\alpha \mid \alpha \in A)$ a collection of elements from K. Then*

$$(z_\alpha \mid \alpha \in A) \text{ is a transcendence basis of } K/k \Leftrightarrow (dz_\alpha \mid \alpha \in A) \text{ is a } K\text{-linear basis of } \Omega^1(K/k).$$

In particular, $\mathrm{td}(K/k) = \mathrm{ldim}_K(\Omega^1(K/k))$.

Proof. $\Rightarrow$ Assume that $(z_\alpha \mid \alpha \in A)$ is a transcendence basis of K/k. To see that the $(dz_\alpha \mid \alpha \in A)$ generates $\Omega^1(K/k)$, it is enough to see that its K-linear span contains all the elements of the form dx for $x \in K$. By assumption and characteristic zero, any such element satisfies a polynomial relation of the form

$$P(x, \overline{z_\alpha}) = 0 \text{ and } \frac{\partial P}{\partial X_0}(x, \overline{z_\alpha}) \neq 0 \text{ where } P \in k[X_0, \dots, X_n] \text{ and } \overline{z_\alpha} = z_{\alpha_1}, \dots, z_{\alpha_n}$$

Equation (1) implies that

$$0 = dP(x, \overline{z_\alpha}) = \frac{\partial P}{\partial X_0}(x, \overline{z_\alpha})dx + \sum_{i=1}^n \frac{\partial P}{\partial X_i}(x, \overline{z_\alpha})dz_{\alpha_i}$$

which shows that dx is in the K-linear span of the $(dz_\alpha \mid \alpha \in A)$. It remains to show that the $(dz_\alpha \mid \alpha \in A)$ are K-linearly independent in $\Omega^1(K/k)$.

Claim. *For every $\alpha \in K$, there is a (unique) derivation $\partial_\alpha \in \mathrm{Der}(K/k)$ on K trivial on k such that*

$$\partial_\alpha(\alpha) = 0 \text{ and } \partial_\alpha(\beta) = 0 \text{ for every } \beta \neq \alpha \in A$$

Proof of the claim. Decompose K/k as

$$k \subset L = k(z_\alpha \mid \alpha \in A) \subset K$$

The existence and uniqueness of ∂_α as a derivation of $\mathrm{Der}(L/k)$ follow from the presentation of L/k as a purely transcendental extension. The claim follows from the fact that any derivation on L extends uniquely to K since K/L is an algebraic extension. □

Now consider a (finite) K-linear relation among the dz_α given as

$$\sum_{\alpha \in A} \lambda_\alpha \cdot dz_\alpha = 0 \text{ where } \lambda_\alpha \in K \text{ all but finitely many are zero}$$

and denote by ∂_α the derivation associated to z_α given by the previous claim. By the universal property of Lemma 5.1, we can find a K-linear form

$$\phi_\alpha : \Omega^1(K/k) \to K$$

with the property that $\phi_\alpha(dz) = \partial_\alpha(z)$ for all $z \in K$. It follows that for all $\beta \in A$,

$$0 = \phi_\beta(\sum_{\alpha \in A} \lambda_\alpha \cdot dz_\alpha) = \sum_{\alpha \in A} \lambda_\alpha \cdot \phi_\beta(dz_\alpha) = \lambda_\beta.$$

Hence, all the coefficients in the linear combination are trivial. We have therefore shown that the $(dz_\alpha \mid \alpha \in A)$ are K-linearly independent and hence form a K-basis of $\Omega^1(K/k)$.

$\Leftarrow$. For the converse, assume first for the sake of contradiction that the z_α are not algebraically independent over k. We can therefore find an algebraic relation of the form

$$P(z_\beta, \overline{z_\alpha}) = 0 \text{ and } \frac{\partial P}{\partial X_0}(z_\beta, \overline{z_\alpha}) \neq 0 \text{ where } P \in k[X_0, \ldots, X_n] \text{ and } \overline{z_\alpha} = z_{\alpha_1}, \ldots, z_{\alpha_n}$$

and the α_i are distinct from β. Using Equation 1, we obtain as previously

$$0 = dP(z_\beta, \overline{z_\alpha}) = \frac{\partial P}{\partial X_0}(z_\beta, \overline{z_\alpha})dz_\beta + \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(z_\beta, \overline{z_\alpha})dz_{\alpha_i}$$

which implies that $\frac{\partial P}{\partial X_0}(z_\beta, \overline{z_\alpha}) = 0$, a contradiction. We have therefore shown that the z_α are algebraically independent over k. To show that they form a transcendence basis of K/k, consider the decomposition

$$k \subset L = k(z_\alpha \mid \alpha \in A) \subset K$$

and assume for the sake of a contradiction that K/L is not an algebraic extension. We can therefore find a derivation ∂ on K which is trivial on L but not on K; say $\partial(x) \neq 0$ for some element $x \in K$. On the one hand, we can write

$$dx = \sum_{\alpha \in A} \lambda_\alpha \cdot dz_\alpha$$

where the right hand side is a finite sum. On the other, we have a K-linear form $\phi : \Omega^1(K/k) \to K$ such that $\phi(dz) = \partial(z)$ for all $z \in K$ by the universal property of Lemma 5.1. We conclude that

$$0 \neq \partial(x) = \phi(dx) = \sum_{\alpha \in A} \lambda_\alpha \cdot \phi(dz_\alpha) = 0$$

which is a contradiction. This finishes the proof of the theorem. □

5.3. **Duality between one-forms and derivations.**

Corollary 5.6 (Second presentation of $\Omega^1(K/k)$)**.** *Assume that K/k is an extension of finite transcendence degree then*

$$\Omega^1(K/k) = \mathrm{Der}(K/k)^* \text{ and } d : x \mapsto (\partial \mapsto \partial(x)).$$

Proof. The map d is well-defined since the evaluation of a derivation at a point is K-linear with respect to the derivation. Furthermore, k-linearly of d is obvious since if $x \in k$ then $\partial(x) = 0$ for all $x \in k$. Finally, for $x, y \in K$ and $\partial \in \mathrm{Der}(K/k)$

$$d(xy)(\partial) = \partial(xy) = \partial(x)y + x\partial(y) = (ydx + xdy)(\partial)$$

and the Leibniz rule follows. It follows by the universal property that

$$\phi : \Omega^1(K/k) \to \mathrm{Der}(K/k)^*$$

sending the differential d on $\Omega^1(K/k)$ to the newly defined d on $\mathrm{Der}(K/k)^*$. Pick $z_1, \ldots, z_n$ a transcendence basis of K/k. By Theorem 5.5,

$$\Omega^1(K/k) = Kdz_1 \oplus \ldots \oplus Kdz_n.$$

Using the claim of the previous theorem, we also obtain n derivations $\partial_1, \ldots, \partial_n$ on K such that $\partial_i(z_j) = 0$ if $i \neq j$ and is equal to 1 if $i = j$. An easy exercise shows that

$$\mathrm{Der}(K/k) = K\partial_1 \oplus \ldots \oplus K\partial_n.$$

Finally, since $dz_i(\partial_j) = \partial_j(z_i) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$, the morphism ϕ sends the basis defined dz_i to the dual of the basis defined by ∂_i. It follows that ϕ is an isomorphism. □

Definition 5.7. Let K/k be a finitely generated extension of fields and $r \geq 1$. A *two-form* ω on K/k is a K-bilinear map

$$\omega : \mathrm{Der}(K/k) \times \mathrm{Der}(K/k) \to K$$

which is alternating in the sense that for every $\partial_1, \partial_2 \in \mathrm{Der}(K/k)$, $\omega(\partial_1, \partial_2) = -\omega(\partial_2, \partial_1)$.

We denote by $\Omega^2(K/k)$ the K-vector space of 2 forms of dimension $\binom{n}{2}$. It is well-known (see any reference in linear algebra) that we have (anti commutative) product denoted

$$\wedge : \Omega^1(K/k) \times \Omega^1(K/k) \to \Omega^2(K/k)$$

given by the formula

$$(\omega_1 \wedge \omega_2)(\partial_1, \partial_2) = \omega_1(\partial_1) \cdot \omega_2(\partial_2) - \omega_1(\partial_2) \cdot \omega_2(\partial_1).$$

Lemma 5.8. *Let K/k be a finitely generated extension of fields and $\omega \in \Omega^1(K/k)$. Then the map*

$$d\omega : (\partial_1, \partial_2) \mapsto \partial_1(\omega(\partial_2)) - \partial_2(\omega(\partial_1)) - \omega([\partial_1, \partial_2])$$

where $[\partial_1, \partial_2] = \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$ is the Lie bracket of derivations in $\mathrm{Der}(K/k)$.

Proof. The definition of $d\omega$ shows that $d\omega$ is alternating. To show that it is bilinear, it is therefore enough to show that it is linear in the first variable. Consider $f \in K$ and $\partial_1, \partial_2 \in \mathrm{Der}(K/k)$.

$$\begin{aligned} d\omega(f\partial_1, \partial_2) &= f\partial_1(\omega(\partial_2)) - \partial_2(\omega(f\partial_1)) - \omega([f\partial_1, \partial_2]) \\ &= f\partial_1(\omega(\partial_2)) - \partial_2(f\omega(\partial_1)) - \omega(f[\partial_1, \partial_2] - \partial_2(f)\partial_1) \\ &= f\partial_1(\omega(\partial_2)) - \Big(f\partial_2(\omega(\partial_1)) + \partial_2(f)\omega(\partial_1)\Big) - \Big(f\omega(\partial_1, \partial_2)) - \partial_2(f)\omega(\partial_1)\Big) \\ &= f\partial_1(\omega(\partial_2)) - f\partial_2(\omega(\partial_1)) - f\omega(\partial_1, \partial_2)) = fd\omega(\partial_1, \partial_2) \end{aligned}$$

The proof of additivity is easier and left as an exercise. □

Definition 5.9. We say that a one-form $\omega \in \Omega^1(K/k)$ is *closed* if the two-form $d\omega \in \Omega^2(K/k)$ given by Lemma 5.8 is equal to zero.

5.4. **An exact sequence.** Let K/k be a field extension of characteristic zero and consider L an intermediate subfield. We construct two morphisms of K-vector spaces.

(1) Viewing $\Omega^1(K/k)$ as an L-vector space, we obtain a morphism of L-vector spaces
$$i_L : \Omega^1(L/k) \to \Omega^1(K/k)$$
obtained by applying Lemma 5.1 to $d_{|L} : L \to \Omega^1(K/k)$.

The usual properties of the tensor product gives an identification
$$\mathrm{Hom}_{L-vect}(\Omega^1(L/k), \Omega^1(K/k)) \simeq \mathrm{Hom}_{K-vect}(\Omega^1(L/k) \otimes_L K, \Omega^1(K/k))$$
which is the (functorial) adjunction between extension and restriction of scalars in commutative algebra. So that the morphism i_l corresponds to a morphism of K-vector spaces:
$$j_L : \Omega^1(L/k) \otimes_L K \to \Omega^1(K/k)$$

(2) Applying Lemma 5.1 to the extension K/k and the morphism $d = d_{K/L} : K \to \Omega^1(K/L)$, we obtain a morphism of K-vector spaces
$$s_L : \Omega^1(K/k) \to \Omega^1(K/L).$$

Corollary 5.10. *With the notation above, the sequence*
$$0 \to \Omega^1(L/k) \otimes_L K \overset{j_L}{\to} \Omega^1(K/k) \overset{s_L}{\to} \Omega^1(L/K) \to 0$$
is a short exact sequence of K-vector spaces

Proof. To see that j_L is injective, it is enough to see that i_L is injective. This follows from Theorem 5.5 and the fact that a transcendence basis of L/k can be completed into a transcendence basis of K/k. Similarly any transcendence basis of K/L can be completed into a transcendence basis of K/k so that s_L is surjective by Theorem 5.5. Finally, the property that $s_L \circ j_L = 0$ follows from the fact
$$s_L(d_{K/k} f) = d_{L/k} f = 0 \text{ for any } f \in L$$
and that $\Omega^1(L/k) \otimes_L K$ is generated as a K-vector space by one-forms of this form. □

5.5. **Geometric interpretation.** Let k be an algebraically closed field. We start by recalling some basic facts about algebraic geometry over an algebraically closed field k and refer to [**?**] for more details. An *affine* algebraic variety X is a Zariski-closed set of k^n for some n. In other words, an affine variety is defined by a (positive) system of the form
$$\begin{cases} P_1(x_1, & \ldots, x_n) = 0 \\ & \vdots \\ P_k(x_1, & \ldots, x_n) = 0 \end{cases}.$$
The Zariski-topology on k^n induces by restriction a *noetherian topology* on X also called the *Zariski-topology.* Hence every affine algebraic variety X is equipped with the structure of a quasi-compact topological space. We say that X is an *irreducible variety* if it is irreducible for this noetherian topology.

Definition 5.11. Let U be an open set of X. We say that a function $f : U \to k$ is *regular around some point* $x \in U$ if there exists a neighborhood V of x in U such that
$$f_{|U}(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} \tag{2}$$
where $f, g \in k[x_1, \ldots, x_n]$ and g does not vanish on V. We say that f is a *regular function on* U and write $f \in \mathcal{O}_X(U)$ if f is regular around every point $x \in U$

Clearly, the regular functions on any given open set U of X form a k-algebra and if $V \subset U$ then we obtain by restriction a morphism of rings $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. It follows that
$$\{\mathcal{O}_X(U) \mid U \subset X\}$$
is an inductive system of rings.

Definition 5.12. Let X be an irreducible affine algebraic variety. The inductive limit $k(X) = \lim_{U \subset X} \mathcal{O}_X(U)$ of this system is called the *field of rational functions on* X.

In other words, a rational function on X is a regular function $f \in \mathcal{O}_X(U)$ on some nonempty open set U of X and two rational functions f, g are equal if they agree on some non empty open subset of X.

Lemma 5.13. *Let X be an irreducible affine variety.*

(i) *The fraction field $k(X)/k$ of X is a finitely generated field extension of k and every finitely generated field extension of k is of this form.*

(ii) *For every $x \in X$, the ring*

$$\mathcal{O}_{X,x} := \{f \in k(X) \mid f \text{ is a regular function around } x\} \subset k(X)$$

is a local ring and the unique maximal m_x is formed by the functions $f \in \mathcal{O}_{X,x}$ satisfying $f(x) = 0$.

Proof. (i). Let $f \neq 0$ be a rational function and let U be an open set such that

$$f_{|U}(x_1, \dots, x_n) = \frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)}$$

as in Equation (2). Denote by V the intersection of U with $U(P) = \{x \in X \mid P(x) \neq 0\}$ which is open and non empty since $f \neq 0$. Set

$$g = \frac{Q(x_1, \dots, x_n)}{P(x_1, \dots, x_n)}$$

which is a regular function on V by definition. Clearly $f \cdot g = 1$ in $\mathcal{O}_X(V)$ so that $f \cdot g = 1 \in k(X)$. It follows that $k(X)$ is a field. Now $X \subset k^n$, the restrictions to X of the coordinates functions $x_1, \dots, x_n$ on k^n are regular functions on X and Equation (2) implies that they generate $k(X)/k$. The converse of (i) follows from the (algebraic) Nullstellensatz (exercise).

(ii) Set $m_x = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$. The fact that $\mathcal{O}_{X,x}$ is a local ring with m_x as the unique maximal ideal follows from the fact that

$$f \in \mathcal{O}_{X,x} \text{ is invertible iff } f \notin m_x.$$

The proof of this easy fact is left as an exercise. □

Our goal is to give a geometric interpretation for

$$\mathrm{Der}(k(X)/k) \text{ and } \Omega^1(k(X)/k).$$

Definition 5.14. Let X be an irreducible affine algebraic variety and $x \in X$. We say that a derivation $\partial \in \mathrm{Der}(k(X)/k)$ is regular at x if $\partial(\mathcal{O}_{X,x}) \subset \mathcal{O}_{X,x}$. We say that a one-form $\omega \in \Omega^1(k(X)/k)$ is regular at x if for every derivation $\partial \in \mathrm{Der}(k(X)/k)$ regular at x, the function $\omega(\partial) \in k(X)$ is regular at x.

It follows immediately from the definitions that if ∂ is regular at any point $x \in U$ then

$$\partial(\mathcal{O}_X(U)) \subset \mathcal{O}_X(U)$$

and similarly that if a one-form ω is regular at every point $x \in U$ and ∂ is a derivation regular on U then $\omega(\partial)$ is a regular function on U. For this reason we denote by $\mathrm{Der}(\mathcal{O}_\mathrm{X}(\mathrm{U})/\mathrm{k})$ and $\Omega^1(\mathcal{O}_X(U)/k)$ the $\mathcal{O}_X(U)$-modules of regular derivations and of regular one-forms on U.

Example 5.15. Set $X = k^n$ so that $k(X) = k(x_1, \dots, x_n)$. If $x_0 \in X$ then

$$\mathcal{O}_{X,x_0} = \{P/Q \mid P, Q \in k[x_1, \dots, x_n] \text{ and } Q(x_0) \neq 0\}$$

It follows easily that the partial derivatives

$$\partial_i : f \mapsto \frac{\partial f}{\partial x_i}$$

are regular derivations at any point $x_0 \in X$. Since they form a $k(X)$-basis of $\mathrm{Der}(k(X)/k)$, any derivation ∂ can be written as

$$\partial = \sum_{i=1}^{n} f_i \cdot \partial_i$$

and ∂ is regular at x if and only if for every i, the function f_i is regular at x. Similarly, the dual basis of $\Omega^1(k(X)/k)$ is given by the one-forms dx_i and any one-form can be written as

$$\omega = \sum_{i=1}^{n} g_i \cdot dx_i$$

which is regular at x iff all the g_i are regular at x.

Definition 5.16. Let X be a Zariski-closed subset of k^n and defined by the vanishing of $f_1, \dots, f_p \in k[x_1, \dots, x_n]$. The *tangent space* TX *of* X is the Zariski-closed subset of k^{2n} given by the vanishing of

$$f_1(x_1, \dots, x_n), \dots, f_s(x_1, \dots, x_n), L_1(x_1, \dots, x_n, \epsilon_1, \dots, \epsilon_n), \dots, L_s(x_1, \dots, x_n, \epsilon_1, \dots, \epsilon_n)$$

where $L_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot \epsilon_j \in k[x_1, \dots, x_n, \epsilon_1, \dots, \epsilon_n]$.

Note the projection $\pi : k^{2n} \to k^n$ restricts to a regular map

$$\pi_X : TX \to X$$

and for any point $p \in X$, the fiber denoted TX_p of π_X over p is called the tangent space of X at p. Since the L_i are linear in the coordinates $\epsilon_1, \dots, \epsilon_p$, TX_p is always a linear subspace of k^n given by the vanishing of

$$L_i(p, \epsilon_1, \dots, \epsilon_n) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(p) \cdot \epsilon_j$$

In particular, π_X is always surjective. We say that X is a *smooth algebraic variety* if the dimension of TX_p does not depend on p.

Definition 5.17. Let X be an irreducible algebraic variety and let U be an open set. An *algebraic vector field* on U is a regular map

$$U \to TX \subset k^{2n}$$

which is a section of π_X. In other words, an algebraic vector field is a tuple $s_1, \dots, s_n \in \mathcal{O}_X(U)$ such that the function

$$v(p) = (p, s_1(p), \dots, s_n(p))$$

takes values in TX. We denote by $\Xi(U)$ the space of regular algebraic vector fields on X.

Construction 5.18. Let X be a Zariski-closed subset of k^n and let U be an open set of X. The coordinates functions $x_1, \dots, x_n$ on k^n define by restrictions regular functions denoted $\overline{x_1}, \dots, \overline{x_n}$ on X and consider

$$\mathrm{Der}(\mathcal{O}_X(U)/k) \to \mathcal{O}_X(U) \times \ldots \times \mathcal{O}_X(U)$$

given by $\partial \mapsto v_\partial = (\partial(\overline{x_1}), \dots, \partial(\overline{x_n}))$.

Proposition 5.19. *With the notation above, the map* $\partial \mapsto v_\partial$ *takes values in* $\Xi(U)$ *and moreover:*

(i) *it is an isomorphism of* $\mathcal{O}_X(U)$*-modules*

$$\mathrm{Der}(\mathcal{O}_X(U)/k) \simeq \Xi(U)$$

(ii) *if* $I \subset \mathcal{O}_X(U)$ *is an ideal and* Z *is the corresponding Zariski-closed subset of* U *then* I *is a differential ideal of* $(\mathcal{O}_X(U), \partial)$ *for* ∂ *if and only if* $v_{\partial|Z} : Z \to TX$ *takes values in* TZ.

Proof. Clearly v_δ is values in $\Xi(U)$ since in $\mathcal{O}_X(U)$, we have for $i = 1, \dots, s$

$$f_i(\overline{x_1}, \dots, \overline{x_s}) = 0$$

which gives after derivation that

$$\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot \partial(\overline{x_j}) = 0$$

for $i = 1, \dots, s$. It follows that the regular function

$$p \mapsto (p, v_1(p), \dots, v_n(p))$$

takes values in TX. To check (i), first note that $\overline{x_1}, \dots, \overline{x_n}$ generate the function field of $\mathcal{O}_X(U)$ and hence that the morphism is injective. It remains to show surjectivity. For that purpose, consider an open set V of k^n such that $V \cap X = U$ so that the restriction morphism

$$\mathcal{O}(V) \to \mathcal{O}_X(U)$$

is surjective and consider $v = (s_1, \dots, s_n)$ an algebraic vector field on X. and pick lifts $t_1, \dots, t_n$ of $s_1, \dots, s_n$ to $\mathcal{O}(V)$. By definition,

$$k[x_1, \dots, x_n] \subset \mathcal{O}(V) \subset k(x_1, \dots, x_n)$$

and hence there is a unique derivation

$$\partial_0 : \mathcal{O}(V) \to \mathcal{O}(V)$$

such that $\partial_0(x_i) = t_i$ and which is trivial on k. To show that it descends to a derivation on $\mathcal{O}_X(U)$, we show that the ideal $(f_1, \dots, f_s)$ is a differential ideal. indeed, if $f = f_i$ then

$$\overline{\partial_0(f)} = \overline{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot t_i} = \sum_{i=1}^{n} \overline{\frac{\partial f}{\partial x_i}} \cdot s_i$$

which is equal to zero in $\mathcal{O}_X(U)$ since v takes values in TX. (ii) is left as an exercise using (i). □

Proposition 5.20. *Let X be an irreducible Zariski-closed subset of k^n and let U be a smooth open set. Denote by TU the inverse image of U under π_X and set*

$$\mathcal{F}_1(U) := \{ f \in \mathcal{O}_{TX}(TU) \mid \text{ for every } p \in U, f_{|T_pU} : T_pU \to k \text{ is linear } \}$$

Then the map

$$\hat{\omega} \mapsto (\partial \mapsto \hat{\omega} \circ v_\partial \in \mathcal{O}_X(U))$$

defines an isomorphism of $\mathcal{O}_X(U)$-modules

$$\mathcal{F}_1(U) \simeq \mathrm{Hom}(\mathrm{Der}(\mathcal{O}_X(U)/k)), \mathcal{O}_X(U)) = \Omega^1(\mathcal{O}_X(U)/k).$$

Proof. Clearly, $\hat{\omega} \mapsto \omega$ is a morphism of $\mathcal{O}_X(U)$-module. To show that it is an isomorphism, we will build the inverse. Let $\omega \in \Omega^1(\mathcal{O}_X(U)/k)$. Since U is smooth, through any point $q \in TU$ we can find a vector field v in a neighborhood of $p = \pi_U(q)$ such that $v(p) = q$. We set

$$\hat{\omega}(q) = \omega(\partial_v)(p)$$

where ∂_v is the derivation associated to the vector field v. Note that if w is another vector field satisfying $w(p) = q$ then

$$(w - v)(p) = 0 \Rightarrow (\delta_v - \delta_w)(m_p) \subset m_p \Rightarrow \omega(\delta_v - \delta_w)(p) = 0$$

using (ii) of the previous proposition and the fact that ω is regular. It follows that the function $\hat{\omega}$ is well-defined. Standard results from algebraic geometry ensures that this is a regular function. This completes the proof of the proposition. □

Definition 5.21. Let $\phi : X \to Y$ be a morphism of algebraic (or analytic) varieties and denote by

$$T\phi = (\phi, d\phi) : TX \to TY.$$

If ω is a regular one-form on Y, the one-form $\phi^*\omega$ on X defined by

$$\widehat{\phi^*\omega} = \widehat{\omega} \circ T\phi$$

is called *the pullback of ω by ϕ.*

5.6. **Frobenius integrability theorem.** We now assume that $k = \mathbb{C}$ and that X is a smooth affine algebraic variety. We say that a regular one-form on X vanishes at p if the linear form $\hat{\omega}(p) : T_pX \to k$ is zero.

Theorem 5.22 (Frobenius integrability theorem for closed forms)**.** *Let ω be a* closed *regular one-form on X which* does not vanish at any point $p \in X$ *and consider the distribution of hyperplanes given by*

$$H : p \mapsto \mathrm{Ker}(\omega(p)) \subset TX_p$$

Given any point $p_0 \in X$, there exists an analytic neighborhood U of p_0 and an analytic submersion $f : U \to \mathbb{C}$ such that the fibers of f are tangent to the distribution H.

Definition 5.23. Let ω be a closed regular one-form on X which does not vanish at any point $p \in X$. We say that two points $x, y \in X$ are *ω-equivalent* if there exists a sequence

$$x_0 = x, x_1, \dots, x_n = y$$

such that for every i, x_i and x_{i+1} lie in a analytic open set U_i equipped with an analytic submersion

$$f_i : U \to \mathbb{C}$$

tangent to the distribution $H(\omega)$ and such that $f_i(x_i) = f_i(x_{i+1})$. An ω-leaf is an equivalence class of this equivalence relation.

Corollary 5.24. *Let* ω *be a* closed *regular one-form on* X *which* does not vanish at any point $p \in X$. *The partition*

$$X = \bigsqcup_{\alpha \in A} L_\alpha$$

into ω*-leaves has the property that for every morphism of irreducible algebraic varieties* $\phi : Y \to X$

$$\phi^* \omega = 0 \text{ iff } \phi(Y) \subset L_\alpha \text{ for some } \alpha \in A$$