LECTURE 4. THE SEIDENBERG EMBEDDING THEOREM

The goal of this lecture is to relate two concepts of solutions for a differential equation

$$(E) : P(y^{(n)}, \dots, y, t) = 0$$

the solutions y in a differential field extension $k/\mathbb{C}(t)$ and the analytic solutions $y(t)$ of the same differential equation.

Notation 4.1. If $y(t)$ in a holomorphic function on some complex domain U, we denote by y the same function seen as an element of the differential field $\mathrm{Frac}(\mathrm{Hol}(U))$ equipped with the derivation

$$g(t) \mapsto g'(t) = \lim_{h \mapsto 0} \frac{g(t+h) - g(t)}{h}.$$

It is called the *differential field of meromorphic functions on* U and denoted $\mathcal{M}(U)$.

4.1. **Analytic solutions of algebraic equations.** Consider an *irreducible* algebraic equation of

$$P(y,t) = a_n(t) \cdot y^n + a_{n-1}(t) \cdot y + \ldots + a_1(t) \cdot y = 0$$

whose coefficients $a_1, \dots, a_n$ are holomorphic functions on some complex domain U. We denote by C the analytic curve of $U \times \mathbb{C}$ defined by the previous equation and consider the (first) projection

$$\pi : C \to U.$$

When looking for a local analytic inverse of π, we make the following observation

- we need to discard the set S of poles of the equations: the times $t \in U$ such that $a_n(t) = 0$. They are the points $t \in U$ over which π has an infinite fiber.
- we also need to discard the ramification points R which are the points $(y,t) \in \mathbb{C} \times U$ satisfying both

$$P(y,t) = 0 \text{ and } P'(y,t) = 0$$

The holomorphic function $\Delta(t) = \mathrm{Res}(P, P')(t)$ is called the *discriminant of the equation.*

Lemma 4.2. *For every* $(y_0, t_0) \in \mathbb{C} \times U$ *such that*

$$P(y_0, t_0) = 0 \text{ and } \Delta(t_0) \neq 0$$

there exists an analytic function $y(t)$ *around* t_0*, satisfying* $y(t_0) = y_0$ *and* $P(y(t), t) = 0$ *for all* t*.*

The proof applies the analytic inversion theorem. Since $\Delta(t) = 0$ admits at most countably many zero, every algebraic equation admits an analytic solution.

4.2. **Analytic solutions of differential equations.** Let U be a complex domain. Consider an irreducible differential polynomial of order n and degree d

$$P(t; y, y', \dots, y^{(n)}) = 0 \text{ with } P \in \mathrm{Hol}(U)[X_0, \dots, X_n]$$

whose coefficients are holomorphic functions on U. In contrast with the convention from differential algebra, we have emphasized the dependence in t in the notation. We write

$$(1) \qquad P(t; X_0, \dots, X_n) = \sum_{i=0}^{d} a_i(t; X_0, \dots, X_{n-1}) \cdot X_n^i$$

so that $a_d(t, X_0, \dots, X_{n-1})$ can be identified with the defining polynomial of *the initial of* P and we write $a_d = i_P(t; X_0, \dots, X_{n-1})$. Instead of a curve C, the equation now defines

$$Z := \{(t, x_0, \dots, x_n) \in U \times \mathbb{C}^{n+1} \mid P(t; x_0, \dots, x_n) = 0\}$$

an analytic subvariety of dimension $n+1$ in $U \times \mathbb{C}^{n+1}$. Instead of the set of poles, we take

$$S := \{(t, x_0, \dots, x_{n-1}) \in U \times \mathbb{C}^n \mid i_P(t, x_0, \dots, x_{n_1}) \neq 0\}$$

and denote by V its complement in $U \times \mathbb{C}^n$. Finally set

$$\pi : Z(P) \cap p^{-1}(V) \to V$$

for the restriction of the projection $p : U \times \mathbb{C}^{n+1} \to U \times \mathbb{C}^n$ on the $n+1$ first coordinates. We make the same observations as previously:

- all fibers of π are finite since we already discarded S
- we still need to get rid of the points of $Z(P) \cap p^{-1}(V)$ where π ramifies.

Lemma 4.3. *With the notation above,*

(i) *π ramifies — that is the differential $d\pi$ of π is not surjective — precisely at the points $(t; x_0, \dots, x_n) \in Z(P) \cap p^{-1}(V)$ where*

$$s_P(t; x_0, \dots, x_n) := \frac{\partial P}{\partial X_n}(t; x_0, \dots, x_n) = 0$$

(iii) *The image of the ramification locus (which is closed by (i)) under π is given by the zero set of*

$$\Delta(P) = \mathrm{Res}(P, s_P) \in \mathrm{Hol}(U)[X_0, \dots, X_{n-1}].$$

Theorem 4.4 (Cauchy-Kovalevskaya)**.** *Consider an irreducible diff. polynomial of order n and degree d*

$$P(t; y, y', \dots, y^{(n)}) = 0 \text{ with } P \in \mathrm{Hol}(U)[X_0, \dots, X_n]$$

and $(t_0, c_0, \dots, c_n) \in U \times \mathbb{C}^{n+1}$ satisfying

$$P(t_0, c_0 \dots, c_n) = 0, i_f(t_0, c_0, \dots, c_n) \neq 0 \text{ and } s_f(t_0, c_0, \dots, c_n) \neq 0.$$

Then:

- (Existence) *there exists a complex disk $\mathbb{D}$ centered at t_0 and $y \in \mathrm{Hol}(\mathbb{D})$ such that*

$$y(t_0) = c_0, \dots, y^{(n)}(t_0) = c_n \text{ and } P(t; y(t), \dots, y^{(n)}(t)) = 0 \text{ for all } t \in \mathbb{D}.$$

- (Uniqueness) *if $(\mathbb{D}_i, y_i)$ for $i = 1, 2$ are two solutions of the previous initial value problem, then*

$$y_{1|\mathbb{D}_1 \cap \mathbb{D}_2} = y_{2|\mathbb{D}_1 \cap \mathbb{D}_2}.$$

4.3. **Seidenberg embedding theorem.** The following theorem makes the connection between analytic geometry and differential algebra. Because of the finiteness assumption in the theorem, it is often used in combination with the compactness theorem from model theory.

Theorem 4.5 (Seidenberg's embedding theorem)**.** *Let k be a countable differential field and l/k a finitely generated extension of differential fields. Assume that we are given a (differential embedding)*

$$i_k : k \to \mathcal{M}(U)$$

for some complex domain U. Then there exists a complex domain $V \subset U$ and an embedding $i_l : l \to \mathcal{M}(V)$ such that the following diagram commutes

$$\begin{array}{ccc} k & \xrightarrow{i_k} & \mathcal{M}(U) \\ \downarrow \subset & & \downarrow \subset \\ l & \xrightarrow[i_l]{} & \mathcal{M}(V) \end{array}$$

Definition 4.6. Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield. We say that a point $x \in U$ is $\mathcal{K}$ *-generic* if it is not the pole nor the zero of any function from $\mathcal{K}$.

Note that by the theorem of isolated zeroes for holomorphic functions, every function from $\mathcal{K}$ has (at most) countably many zeroes and poles in U. Since $\mathcal{K}$ is countable and that a countable union of countable sets is countable, it follows that $\mathcal{K}$-generic points do exist.

Lemma 4.7. *Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield and let $x \in U$ be a $\mathcal{K}$-generic point. The evaluation*

$$\mathrm{ev}_x : \mathcal{K} \to \mathbb{C}$$

identifies $\mathcal{K}$ with a countable subfield $\mathcal{K}(x)$ of $\mathbb{C}$.

Proposition 4.8. *Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield and let $x \in U$ be a $\mathcal{K}$-generic point. Let f be an holomorphic function in a neighborhood of x and assume that*

$$f(x), f'(x), \dots, f^{(n)}(x)$$

are algebraically independent over $\mathcal{K}(x)$ in $\mathbb{C}$. Then f satisfies no algebraic differential equation with order $\leq n$ and parameters from $\mathcal{K}$.

Proof. Otherwise, the function f satisfies a nontrivial algebraic differential equation

$$P(y, y', \dots, y^{(n)}) = 0$$

where $P \in \mathcal{M}(U)[X_0, \dots, X_n]$ can be written as

$$P = \sum a_{i_0,\dots,i_n}(t) \cdot X_0^{i_0} \cdots X_n^{i_n}$$

Since x is $\mathcal{K}$-generic, the coefficients of P can be evaluated at x and the previous polynomial relation implies that

$$\sum a_{i_0,\dots,i_n}(x) \cdot f(x)^{i_0} \cdots (f(x)^{(n)})^{i_n} = 0$$

which contradicts our assumption that $f(x), \dots, f(x)^{(n)}$ are algebraically independent over $\mathcal{K}(x)$. □

Corollary 4.9. *Let $\mathcal{K}$ be a countable differential subfield of $\mathcal{M}(U)$. Then $\mathcal{M}(U)$ has infinite differential transcendence degree over $\mathcal{K}$.*

Proof of Seidenberg theorem. Let $\mathcal{K}$ be a countable differential field and $L/\mathcal{K}$ a finitely generated extension of differential fields. Assume that we are given a (differential embedding)

$$i_{\mathcal{K}} : \mathcal{K} \to \mathcal{M}(U)$$

for some complex domain U. Clearly, by induction, it suffices to see that for any differential field extension $L = \mathcal{K}(\xi)/\mathcal{K}$ generated by a single element can be embedded inside a field of meromorphic function.

- Assume that ξ is differentially transcendental over $\mathcal{K}$. We use the previous corollary to find a function $g \in \mathcal{M}(U)$ which is differentially transcendental over $\mathcal{K}$ and extend $i_{\mathcal{K}}$ to
$$i_L : L \simeq \mathcal{K}(g) \subset \mathcal{M}(U).$$
- Assume that ξ is differentially algebraic over $\mathcal{K}$ and denote by $P \in \mathcal{K}\{X\}$ the (irreducible) differential polynomial of minimal order and degree such that
$$(E) : P(t; y, y', \dots, y^{(n)}) = 0.$$

The description of prime differential ideals of $\mathcal{K}\{X\}$ shows that it is enough to find a function $g \in \mathcal{M}(V)$ which satisfies (E) and no other differential equation of minimal order. To do so consider, $t_0 \in U$ which is $\mathcal{K}$-generic and $c_0 \dots, c_{n-1} \in \mathbb{C}$ algebraically independent over $\mathcal{K}(x)$. Note that by algebraic independence,

$$i_f(t_0, c_0, \dots, c_{n-1}) \neq 0$$

and therefore there exists $c_n \in \mathbb{C}$ such that $P(t_0, c_0, \dots, c_{n-1}) = 0$. Furthermore, note that

$$s_f(t_0, c_0, \dots, c_n) \neq 0$$

since $\Delta(P)(c_0, \dots, c_{n-1}) \neq 0$ again by algebraic independence. Applying Cauchy-Kovalevskaya Theorem and the previous corollary finishes the proof. □