Galois groups of non-standard curves in CCM

Rémi Jaoui

University of Notre Dame, Model theory seminar

November 17th

- The subject of my talk is the structure of binding groups (or Galois groups) in the theory **CCM** of compact complex manifolds.
- My talk will be based on an example of a paper with L. Jimenez and A. Pillay and more recent discussions with Anand based on classical constructions from complex geometry.

Binding groups: Let $T = T^{eq}$ be a stable theory, $\mathbb{M}$ a monster model. Assume that T interprets a pure algebraically closed field D_0 over $\emptyset$.

- Let D be A-definable set. D is internal to D_0 if D is in definable bijection (involving additional parameters) with a definable subset of D_0^n.
- The binding group $G(D)$ of D is the group of permutations of D induced by elementary permutations of $\mathbb{M}$ fixing A and D_0 pointwise. It is a stunning consequence of geometric stability theory that:

  $G(D)$ is always definable and definably isomorphic to an algebraic group.

- Under some mild assumptions on D, $G(D)$ acts transitively on the definable set D. In that case, we have a Galois correspondence and we will say that $G(D)$ is the Galois group of D (over A relatively to D_0).

Theorem (Kovacic, Singer, ...)

For every connected algebraic group G, there exists a definable set D internal to the constants such that the binding group of D is definably isomorphic to G.

- They also describe the dependence in the set of parameters A over which D is defined. Note that if D is defined over A and $A \subset B$ then

$$i : G(D/B) \to G(D/A)$$

  is an injection but not always surjective. For example, if B is a model then $G(D/B) = 0$.
- If we restrict to strongly minimal sets D, there are only few possibilities for $G(D)$:
  - either $G(D)$ is finite
  - or $G(D)$ in abelian by finite. In that case, the connected component is

$$G^{0} = \mathbb{G}_m, \mathbb{G}_a, \text{or an elliptic curve } E$$

  - or $G(D) = Aff_2$ or $G(D) = PSL_2$.

Goal of my talk: show that (in the connected case), all the groups above occur as Galois groups of non-standard curves in **CCM**.

Setting (fixed one for all):

- $\mathcal{A}$ is the standard model of **CCM**: the sorts are (connected) compact complex manifolds. If $X_1, \ldots, X_n$ are CCM, we name every closed analytic subset

$$Z \subset X_1 \times \ldots \times X_n$$

  by a predicate.

- $\mathcal{A}$ admits the elimination of quantifiers and imaginaries in this language.
- We fix $\mathcal{A} \preccurlyeq \mathcal{A}^*$ is a saturated elementary extension of the standard model of **CCM**.

Standard model $\mathcal{A}$	Elementary extension $\mathcal{A}^*$
Y a (connected) compact complex manifold.	$\bar{y} \in \mathcal{A}^*$ a realization of the generic type $p_Y \in S(\emptyset)$ of Y
$f : X \longrightarrow Y$ an holomorphic fibration definable in CCM	"the" generic fibre $X_{\bar{y}} = f^{-1}(\bar{y})$ over a realization $\bar{y}$ of p_Y

- If X is compact then an holomorphic fibration $f : X \to Y$ is always definable in **CCM**.
- More generally, if X is Zariski-open set in a compact complex manifold $\hat{X}$ (i.e. the complement $\hat{X} \setminus X$ is a closed analytic subset) and f extends to a meromorphic morphism $\hat{f} : \hat{X} \dashrightarrow Y$ then the holomorphic fibration $f : X \to Y$ is definable in **CCM**.

Questions: Let $f : X \to Y$ be an holomorphic fibration definable in **CCM** as above.

(1) When is the generic fibre of $f : X \to Y$ internal to the projective line $\mathbb{P}^1$?

(2) When it is the case, let G be the binding group of the generic fibre. The group G is a natural invariant of the fibration, what does it encode?

(3) Describe some explicit examples, where this Galois group can be computed.

I will describe some partial answers to these questions based on the algebraic properties of the base Y and of the total space X.

The algebraic dimension $a(X)$

Let X be a compact complex manifold.

Definition

The algebraic dimension of X denoted $a(X)$ is the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(X)$ of meromorphic functions on X.

Equivalently,

- $a(X)$ is the maximal dimension of a projective manifold X such that there exists a dominant meromorphic morphism

$$\phi : X \dashrightarrow P.$$

- Model theoretically, $a(X)$ is the Morley rank (for example) of the biggest internal quotient of (the generic type of) X.

Classical theory to study $a(X)$ (Kodaira): based on the group $Pic(X) \simeq H^1(X, \mathcal{O}_X^*)$ of line bundles on X and the intersection product

$$\begin{array}{rcl} Pic(X) \times Pic(X) \to & H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to & H^4(X,\mathbb{Z}) \simeq \mathbb{Z} \\ (\mathcal{L},\mathcal{L}') \longrightarrow & (c_1(\mathcal{L}), c_1(\mathcal{L}')) \longrightarrow & c_1(\mathcal{L}) \wedge c_1(\mathcal{L}') \end{array}$$

Definition

We say that a compact complex manifold X is algebraic (or Moishezon) if X has maximal algebraic dimension i.e.

$$dim(X) = a(X).$$

- So X is algebraic if and only if there exists a dominant meromorphic morphism
$$\phi : X \dashrightarrow P_1$$
with $dim(X) = dim(P_1)$ and P_1 projective. Note that ϕ has a finite generic fibre.
- **Moishezon Theorem**: If X is algebraic then there exists a projective variety P_2 such that
$$P_2 \to X \dashrightarrow P_1$$
and $P_2 \to X$ is bimeromorphic.
- If $dim(X) = 2$ then X is a projective manifold (Kodaira) but it is not necessarily true in higher dimensions (Hironaka example).

Binding groups as an obstruction to algebraicity

Proposition

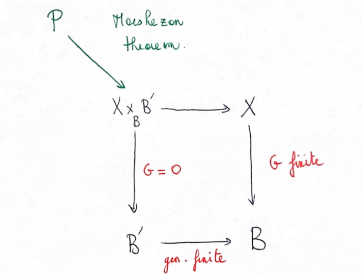
Let $f : X \to B$ be a fibration of CCM. Assume that ***the generic fibre is internal to the constants*** *and that* ***the base B is algebraic****. TFAE*

(i) *The binding group G of the generic fibre is trivial.*

(ii) *The binding group G of the generic fibre is finite.*

(iii) *X is algebraic*

- The implication $(i) \Rightarrow (iii)$ follows from Jimenez's theory of uniform relative internality: if $G = 0$, then the fibration is uniformly internal hence preserves internality to $\mathbb{P}^1$.
- Everything else follows from Moishezon's Theorem:



A compact complex manifold X is called Kahler if it admits a hermitian metric such that the associated $(1,1)$-form is closed.

- **A geometric consequence:** A Kahler manifold X satisfies Hodge decomposition. In particular, the first Betti number

$$b_1(X) = H_1(X, \mathbb{Z}) \text{ is even.}$$

- **A model-theoretic consequence:** (Moosa) A Kahler manifold X is essentially saturated: there exists a countable relational language $\mathcal{L}_0$ formed by analytic subsets of X^n such that every analytic subset of X^n is $\mathcal{L}_0$-definable with parameters.

Definition (Fujiki)

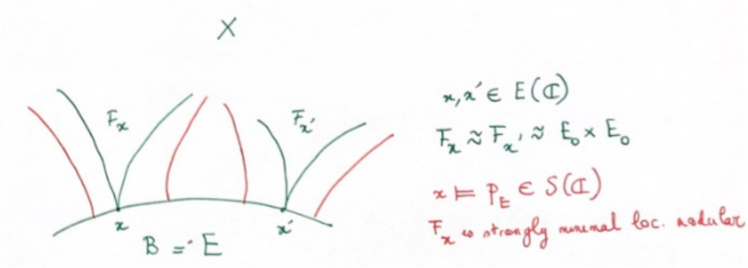
The class $\mathcal{C} \subset \mathcal{A}$ is the class of compact complex manifolds which are meromorphic images of Kahler manifolds.

Every algebraic compact manifold is in the class $\mathcal{C}$. Essential saturation holds more generally for compact complex manifolds in the class $\mathcal{C}$.

A beautiful (but scary) example

Building on a construction of Lieberman, Pillay and Scanlon constructed a complex compact manifold X **of dimension** 3 **outside of** $\mathcal{C}$ and a surjective holomorphic morphism $f : X \to E$ towards an elliptic curve E such that:

- The family looks isotrivial: all the complex (i.e. standard) fibres of f isomorphic to the same complex Abelian variety A_0 (in fact, A_0 is a product of two elliptic curves)
- But isn't...: the (non-standard) generic fibre of f is a strongly minimal (non-standard) locally modular group.



So the behavior of a general fibre — i.e. the behavior of "most" standard fibres — does not (necessarily) reflect the behavior of the generic fibre

Let $f : X \to Y$ be a fibration of relative dimension one. The generic fibre X_y is a non-standard curve.

Theorem (Moosa)

- *(weak form)* X_y *is in definable bijection (involving possible parameters) with a definable subset* $D \subset \mathbb{P}^1(\mathcal{A}^*) \times \ldots \mathbb{P}^1(\mathcal{A}^*)$.
- *(strong form)* X_y *is definably isomorphic (involving possible parameters) to a non-standard projective curve.*

Proof: weak form in the class $\mathcal{C}$.

Assume X is in the class $\mathcal{C}$. Then the 3-sorted structure $(X, Y, \mathbb{P}^1)$ is essentially saturated. We work in a countable language $\mathcal{L}_0$:

- The fibration $f : X \to Y$ is defined over a finite set A_0.
- Pick a non-standard realization y of the generic type of Y. $tp(y/M)$ is the non-forking extension of the generic type $p_0 = tp(y/A_0)$.
- By saturation of the standard model, pick $a_0 \models p_0$ in the standard model.
- Apply the standard Riemann existence theorem to $f^{-1}(a_0)$.

□

Let $f : X \to Y$ be a fibration of relative dimension one. By Rahim's theorem, the generic fibre is definably isomorphic to a (non-standard) projective curve P. Denote by $g \geq 0$ the genus of P.

Corollary

Let $f : X \to Y$ be a fibration of relative dimension one. If $g \geq 2$ then the Galois group of the generic fibre is finite. In particular, if the base Y is algebraic, then X is algebraic.

Denote by G the binding group of the generic fibre.

- The strong form of Rahim's theorem implies that G can be identified with a subgroup of $Aut(P)$.
- P is smooth and has genus $g \geq 2$ so that $Aut(P)$ (and therefore G) is finite.

Conclusion for our purposes: there are two interesting cases to produce non-trivial connected Galois groups:

$$g = 0 \text{ and } g = 1.$$

Consider E a complex elliptic curve. The universal covering of E is $\mathbb{C}$ and there exists a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ such that

$$E = \mathbb{C}/\Lambda.$$

We consider successive quotients:

$$\mathbb{C} \to \mathbb{C}/\omega_2\mathbb{Z} \to \mathbb{C}/\Lambda.$$

Notice that $\mathbb{C}/\omega_2\mathbb{Z} \simeq \mathbb{C}^*$ through $z \mapsto exp(\frac{2i\pi}{\omega_2}z)$ and that the image of the lattice Λ is the cyclic group generated by $q = exp(2i\pi\frac{\omega_1}{\omega_2})$.

Lemma

Every complex elliptic curve E can be presented as

$$\mathbb{C}^*/q^{\mathbb{Z}}$$

where $q = exp(2i\pi\tau)$, $\tau = \omega_1/\omega_2$ and ω_1 and ω_2 are a basis of periods of E.

$\tau \notin \mathbb{R}$ and we can always assume that $Im(\tau) > 0$ so that $|q| < 1$.

Fix $q \in \mathbb{C}$ with $|q| < 1$ and E_q the corresponding elliptic curve.

- We consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ given by:
$$n.(x,y) = (q^n x, q^n y).$$
- The discrete group $\mathbb{Z}$ acts properly discontinously without fixed points on $\mathbb{C}^2 \setminus \{(0,0)\}$ so the quotient
$$H_q = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}$$
exists as a complex manifold. The complex manifold H_q is compact.
- The canonical morphism $\pi : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{P}^1$ is equivariant under this action so we obtain
$$\pi : H_q \to \mathbb{P}^1.$$
- This is an elliptic fibration and all the complex fibres are isomorphic to the same elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$.

H_q is called **the elliptic Hopf surface associated to** q. We will see that it is not Kahler (Rahim showed that it is not even essentially saturated).

Proposition

Let $q \in \mathbb{C}$ with $|q| < 1$. Consider the associated elliptic Hopf surface:

$$\pi : H_q \to \mathbb{P}^1.$$

Denote by G the Galois group of the generic fibre. Then $G = E_q$.

Key idea: H_q is not merely an elliptic fibration but an elliptic principal bundle: we have a natural action

$$E_q \times H_q \to H_q$$

which preserves the fibres and such that E_q acts sharply transitively on each fibre.

- **Consequence:** the generic fibre is internal to the constants and the Galois group G is definably isomorphic to a definable subgroup of $E_q(\mathcal{A}^*)$.
- Since the base is algebraic, either $G = E_q(\mathcal{A}^*)$ or G is trivial and H_q is an algebraic surface.
- As a real manifold, we have

$$H_q \simeq H_{q'} \simeq \mathbb{S}^1 \times \mathbb{S}^3.$$

  So H_q is not a Kahler manifold and $a(H_q) \neq 2$.

Kodaira's theorem

Theorem (Kodaira)

Let X be a compact complex surface. If $a(X) = 1$, then X is an elliptic fibration: there exists a holomorphic fibration:

$$\phi : X \to C$$

toward a projective curve C and the fibres are elliptic curves.

- This was generalized by Ueno, Fujiki and Campana for arbitrary compact complex manifolds of dimension n and algebraic dimension $n - 1$.
- The original proof is based on the positivity properties of line bundles. I don't know a model theoretic proof excluding the case $g = 0$.

Corollary

Let $f : X \to Y$ be a fibration of CCM of relative dimension one with an algebraic base Y. Then the Galois group G of the generic fibre of f is either trivial or the connected component of G is an elliptic curve.

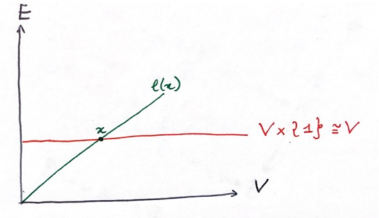
To obtain non-trivial examples with $g = 0$, we now consider the opposite case: fibrations over a base Y with $a(Y) = 0$.

Linear setting: Consider X a compact complex manifold, V a vector bundle over X of rank n. We have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{P}(V) \\ \downarrow & \swarrow & \\ X & & \end{array}$$

Although not compact, the vector bundle V is definable in CCM as

$V \simeq \mathbb{P}(V \oplus E) \setminus \mathbb{P}(V)$ where E is the trivial line bundle.



The generic fibres of V and $\mathbb{P}(V)$ are both internal to the constants. Denote by $G(V)$ the Galois group of the generic fibre of V and $G(\mathbb{P}(V))$ the Galois group of the projectivization.

Two remarks on the linear setting

- $G(V)$ and $G(\mathbb{P}(V))$ are definably isomorphic respectively to algebraic subgroups of GL_n and PGL_n and we have an exact sequence

$$1 \to G(V) \cap \mathbb{G}_m \to G(V) \to G(\mathbb{P}(V)) \to 1.$$

- This construction is trivial over an algebraic base X:

Lemma

Let X be a projective variety. For every vector bundle V over X,

$$G(V) = G(\mathbb{P}(V)) = 0.$$

By GAGA, if V is an analytic vector bundle on X then V (and its total space) is algebraic. So $G(V) = 0$.

- If $a(X) = 0$ then a prime model $\mathcal{A}_X$ of **CCM** over a generic point of X "has no new constants":

$$\mathbb{P}^1(\mathcal{A}_X) = \mathbb{P}^1(\mathbb{C}).$$

So $G(V)$ and $G(\mathbb{P}(V))$ are definably isomorphic to complex algebraic groups.

Definition

A (complex) K3-surface X is a compact complex surface which admits a global nowhere vanishing 2-form $\omega \in \Omega^2_X$ (i.e. $\Omega^2_X \simeq \mathcal{O}_X$) and with irregularity 0:

$$H^1(X, \mathcal{O}_X) = \{0\}.$$

Using the exponential exact sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$, we obtain:

$$H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).$$

(1) $H^1(X, \mathbb{Z}) = 0$. In fact, a K3-surface X is simply connected.

(2) Hodge decomposition implies that for any(complex) K3-surface,

$$H^0(X, \Omega^1_X) = H^0(X, \Theta_X) = \{0\}.$$

(3) For a K3-surface, $Pic(X)$ is a finitely generated abelian group of rank

$$r = \rho(X) \in \{0, \ldots, 20\}.$$

Complex K3-surfaces X can have arbitrary algebraic dimension $a(X)$, arbitrary position with respect to Zilber trichotomy and arbitrary Picard number $\rho(X) \in \{0, \ldots, 20\}$.

Definition

A "generic" K3-surface is a K3-surface X with $Pic(X) = 0$.

A "generic" K3-surface X satisfies $a(X) = 0$. In fact, X is strongly minimal and disintegrated (but in general not ω-categorical).
This can be seen directly but also follows from the same program I used to describe generic planar algebraic vector fields using what follows.

Theorem (See Compact complex surfaces, Van de Ven, Peters, Barth)

Let X be any K3-surface. Then there exists a polydisk $\Delta \subset \mathbb{C}^n$ centered at 0 and a (flat) family of K3-surfaces $\pi : \mathcal{X} \to \Delta$ such that:

- $\mathcal{X}_0 \simeq X$,
- *Outside of a countable union of proper analytic subset $Z_i \subset \Delta$:*

$$\text{for every } s \in \Delta \setminus \cup_{i \in \mathbb{N}} Z_i, \mathcal{X}_s \text{ is K3-surface with } Pic(X_s) = 0.$$

A criterion for solvability of the Galois group

Proposition

Let X be a compact complex manifold and V a vector bundle over X (or more generally a coherent sheaf over X). Then the following are equivalent:

(i) *The connected component of the Galois group $G(V)$ of V is solvable.*

(ii) *There exists a base change $f : X' \to X$ with $dim(X) = dim(X')$ such that the vector bundle f^*V is filtrable: there exists a finite sequence of coherent sheaves on X'*

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_n = f^*V$$

with $rk(\mathcal{F}_{i+1}/\mathcal{F}_i) = 1$.

Two ingredients:

- Lie-Kolchin Theorem: A solvable connected linear algebraic group can be embedded in the group of triangular matrices.
- If $\phi : X \dashrightarrow X'$ is bimeromorphic then ϕ^*V is filtrable if and only if V is filtrable.

Corollary

Let X be a "generic" $K3$-surface. Then the Galois group $G(\mathbb{P}(T_X))$ of the generic fibre of $\pi : \mathbb{P}(T_X) \to X$ is (definably isomorphic to) PSL_2.

- The strong form of the non-standard Riemann existence theorem, we first prove that $G(\mathbb{P}(T_X))$ is definably isomorphic to an algebraic subgroup PSL_2.
- By contradiction: Assume that $G \neq PSL_2$. Then the connected component of G^0 is solvable. The exact sequence:

  $$1 \to \mathbb{G}_m \cap G(T_X) \to G(T_X) \to G(\mathbb{P}(T_X)) \to 1$$

  implies that the connected component of $G(T_X)$ is solvable too!
- Let $\pi : X' \to X$ be a base change. The ramification locus has **pure** codimension one but X has no curves... So π is unramified and since X is simply connected, π is bimeromorphic.
- So T_X is filtrable and there exists a subline bundle $\mathcal{L} \subset T_X$ (i.e. an analytic foliation on X). But $Pic(X) = 0$ so that $\mathcal{L} = \mathcal{O}_X$.

This implies that X supports a global vector field, contradiction.

Theorem

Let G be a connected complex algebraic group acting faithfully and transitively on a strongly minimal set, i.e.

$$G = \mathbb{G}_m, \mathbb{G}_a, \text{ a complex elliptic curve } E, G = Aff_2 \text{ and } G = PSL_2.$$

Then G can be realized as the Galois group of a non-standard curve in ***CCM****.*

(a) If G is an elliptic curve, use elliptic Hopf surfaces over $\mathbb{P}^1$.

By Kodaira theorem, one can not obtain non-trivial linear algebraic groups over $\mathbb{P}^1$ (and more generally over any algebraic base)

(b) If $G = PSL_2$, use the projectivization of the tangent bundle of a "generic" K3-surface.

(c) The other linear algebraic groups $Aff_2, \mathbb{G}_m, \mathbb{G}_a$ are all subgroups of PSL_2. One can conclude using (b) and the Galois correspondence.

Thank you for your attention!