A model-theoretic invitation to web geometry

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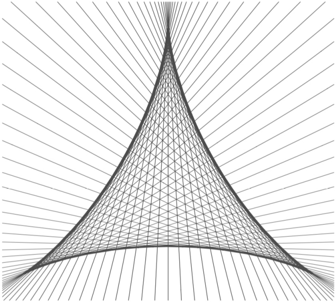
- The title is the reference to the book

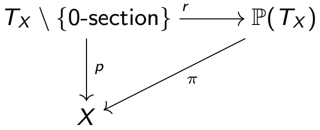
  An invitation to web geometry by L. Pirio and J. V. Pereira.
- Date of birth of web geometry: Spring 1927 in Naples.



A 3-web on a hyperbolic triangle Δ: from every point $x \in \Delta$, a set of three lines $\{l_1(x), l_2(x), l_3(x)\}$ of $T\Delta_x$.

The projectivized tangent bundle

We fix X a **smooth irreducible complex algebraic variety of dimension two**. T_X denotes the tangent space of X and $\mathbb{P}(T_X)$ the space of lines of T_X.

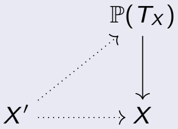


Definition

A **foliation** is a rational section of $\pi : \mathbb{P}(T_X) \to X$: the germ of a definable section of π at the generic point of X.

$$\sigma : X \dashrightarrow \mathbb{P}(T_X) \text{ satisfying } \pi \circ \sigma = id_X.$$

An **irreducible web** is an algebraic section of $\pi : \mathbb{P}(T_X) \to X$: a rational section after a generically finite extension $\phi : X' \dashrightarrow X$:



Typical statement: Given an additional structure on X (a vector field $v : X \to T_X$ or a rational dominant $\sigma : X \dashrightarrow X$), describe the relations between:

- the properties of the differential equation (X, v) or of the difference (X, σ) (seen respectively in $\mathbf{DCF}_0$ and **ACFA**).
- the structure of algebraic sections of the projection $\mathbb{P}(T_X)$ compatible with this additional structure (the so-called **invariant foliations and invariant webs**).

Plan of my talk:

(1) Describe an "improved formalism" for foliations and webs in dimension two.

(2) Geometric stability theory in $\mathbf{DCF}_0$ and comparisons with **CCM** and $\mathbf{ACFA}_0$.

(3) A Galois-theoretic analysis of invariant webs.

The theory of foliations and webs can be carried out purely analytically and applied in **CCM**. For presentation purposes, I will describe webs and foliations only for an algebraic variety X.

An example

Intuitive idea: Having many algebraic sections is a particular instance of having "a lot of structure".

- ($T = \mathbf{ACF}_0$) If $\pi : E \to X$ be a vector bundle of rank n over X then E is generated by its rational sections: there are rational sections $\sigma, \dots \sigma_n$ of π such that

$$(\sigma_1(x), \dots \sigma_n(x)) \text{ form a basis of } E_x \text{ for every } x \in U \subset X.$$

In particular, $E = T_X$, any algebraic variety X admits many foliations and webs.

Corollary (GAGA, $T = \mathbf{CCM}$)

Let M be a compact complex manifold. If there exists an analytic vector bundle E over M without any non-zero meromorphic section then M is not isomorphic to a projective algebraic variety.

Proposition (Prototype statement, $T = \mathbf{CCM}$)

Let M be a compact complex ***surface****. Assume that M does not support any analytic web then the generic type of M is minimal and locally modular.*

Horizontal divisors of $\mathbb{P}(T_X)$

Consider an algebraic section:

$$\begin{array}{ccc} & & \mathbb{P}(T_X) \\ & \nearrow & \downarrow \\ X' & \dashrightarrow & X \end{array}$$

$\overline{\sigma(X')}$ is a closed irreducible hypersurface of $\mathbb{P}(T_X)$ which projects dominantly on X.

Definition

The group $Div_h(\mathbb{P}(T_X))$ of horizontal divisors is the free abelian group generated by the closed irreducible hypersurfaces of $\mathbb{P}(T_X)$ which dominate X.

- A non-horizontal hypersurface has the form $\pi^{-1}(C)$ for some irreducible curve $C \subset X$. So if $U \subset X$ is a dense open set then

$$Div_h(\mathbb{P}(T_X)) \to Div_h(\mathbb{P}(T_U))$$

  is an isomorphism.
- If $D = k_1 Z_1 + \ldots k_n Z_n$ and r_i the cardinal of the generic fibre

$$\pi_{|Z_i} : Z_i \to X$$

  we set $deg(D) = \sum r_i k_i \in \mathbb{Z}$.

Definition

Let $r \geq 1$. An algebraic r-web on X is an effective horizontal divisor

$$D = k_1 Z_1 + \dots k_n Z_n \in Div_h(\mathbb{P}(T_X)) \text{ (with } k_i > 0)$$

of degree r.

- **Case** $r = 1$: An algebraic 1-web is always irreducible and of the form $\overline{\sigma_{\mathcal{F}}(X)}$ for some foliation $\sigma_{\mathcal{F}} : X \dashrightarrow \mathbb{P}(T_X)$.
- An algebraic r-web is called reduced if $k_i = 1$ for all i. A reduced algebraic r-web can be identified with its support $\mid D \mid = Z_1 \cup \dots \cup Z_n$. Then its degree is the cardinal of the generic fibre of

$$\pi : \mid D \mid = Z_1 \cup \dots \cup Z_n \to X.$$

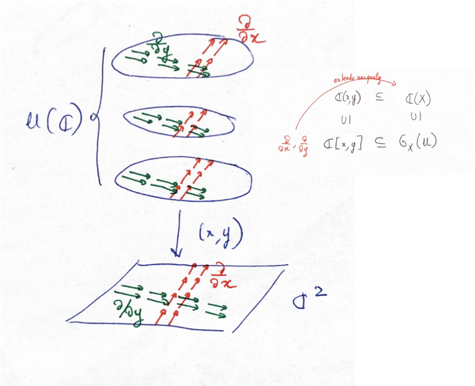
- To formulate a second description of webs, I will need to describe explicitly the sheaf $\mathcal{O}_{T_X}$ of functions on T_X.

Open covers with étale coordinates

An **affine open set with étale coordinates** of X is an affine open set $U \subset X$ endowed with two functions $x, y \in \mathcal{O}_X(U)$ such that:

$$(x, y) : U \to \mathbb{A}^2$$

is étale, i.e. a local analytic diffeomorphism at each of its complex points.



Any smooth algebraic surface X can be covered by affine open sets with étale coordinates.

Open covers with étale coordinates

Let U an open set with étale coordinates $x, y \in \mathcal{O}_X(U)$

- Any vector field v on U and any one-form $\omega \in \Omega^1_X(U)$ can be uniquely written as:

$$v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \text{ and } \omega = a.dx + b.dy \text{ with } a, b \in \mathcal{O}_X(U).$$

- The differential $d : \mathcal{O}_X(U) \to \Omega^1_X(U)$ is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

  for $f \in \mathcal{O}_X(U)$.

So differential calculus on U is simply an extension of differential calculus on $\mathbb{C}^2$ with $k[x, y]$ replaced by $\mathcal{O}_X(U)$.

Let U is an affine open sets with étale coordinates $x, y \in \mathcal{O}_X(U)$.

- We form the algebra $\mathcal{O}_X(U)[dx, dy]$ of formal polynomials with coefficients in $\mathcal{O}_X(U)$ and formal variables dx and dy.
- Example:

$$\omega = f_1.dx.(dy)^2 + f_2.(dx)^3 + \ldots + f_3.dx(dy)^4$$

Lemma

Any element of $\mathcal{O}_X(U)[dx, dy]$ defines naturally a function on $p^{-1}(U)$ and this induces an isomorphism:

$$\mathcal{O}_X(U)[dx, dy] \simeq \mathcal{O}_{T_X}(p^{-1}(U))$$

- $\mathcal{O}_X(U)[dx, dy]$ is a graded algebra graded by the total degree in dx and dy. A coordinate-free presentation of $\mathcal{O}_X(U)[dx, dy] \simeq Sym^\bullet(\Omega^1_X(U))$
- These graded local algebra come together as a global sheaf of graded algebras on X denoted by

$$Sym^\bullet(\Omega^1_X) = \mathcal{O}_X \oplus \Omega^1_X \oplus \ldots \oplus Sym^k(\Omega^1_X) \oplus ...$$

Local equations of a web

Let W be an algebraic web on X. Consider

$$Z = \overline{r^{-1}(W)} = r^{-1}(W) \cup \{0 - \text{section}\}$$

as an hypersurface of T_X.

- If U an affine open sets with étale coordinates (x, y) then $p^{-1}(U)$ is affine and:

  $$Z \cap p^{-1}(U) := (\omega = 0)$$

  for some function $\omega \in \mathcal{O}_{T_X}(p^{-1}(U)) \simeq \mathcal{O}_X(U)[dx, dy]$.
- If W has degree r then f is an homogeneous polynomial of degree r:

  $$\omega = f_0(dx)^r + f_1 dx(dy)^{r-1} + \ldots + f_r(dy)^r$$
- If U' is another open set and $\omega' = 0$ is another equation for W then is a equation of generates then

  $$\omega'_{U' \cap U} = f.\omega_{U' \cap U} \text{ for some } f \in \mathcal{O}^*_X(U \cap U').$$

Conclusion: the local equations of the divisor glue together and define a global invertible subsheaf $\mathcal{L} \subset \mathrm{Sym}^k(\Omega^1_X)$.

Proposition

Let X be a smooth irreducible algebraic surface. There is a one to one correspondence between the sets of

(i) *effective irreducible divisors W of degree r in $Div_h(\mathbb{P}(T_X))$.*

(ii) *invertible subsheaves $\mathcal{L} \subset Sym^r(\Omega^1_X)$ such that $Sym^r(\Omega^1_X)/\mathcal{L}$ does not have torsion.*

- Indeed to show that $Sym^r(\Omega^1_X)/\mathcal{L}$ does not have torsion we need to show that:
  $f.\omega$ vanishes on W for some $f \in \mathcal{O}_X(U) \Rightarrow \omega$ vanishes on W
- $(f = 0)$ defines an algebraic curve C on U and since W is horizontal:

$$W \setminus \pi^{-1}(C) \text{ is Zariski-dense in } W.$$

- ω vanishes on $W \setminus \pi^{-1}(C)$ so it vanishes on W.

Operations on webs

Let $\mathcal{W}, \mathcal{W}'$ be two algebraic webs on X represented by horizontal divisors W and W' in $\mathbb{P}(T_X)$.

(i) There is a sum operations on webs on X denoted respectively

$$\mathcal{W} \boxtimes \mathcal{W}' \text{ and } W + W'.$$

(ii) If $\phi : X' \dashrightarrow X$ is dominant and generically finite then there is a pull-back operation

$$\mathcal{W} \mapsto \phi^* \mathcal{W}$$

which preserves the degrees of webs.

(iii) If G is a finite group acting on an affine variety X and $\{\mathcal{W}_1, \ldots, \mathcal{W}_n\}$ denotes the orbit of $\mathcal{W}$ under G, there is a unique web W_Y on $Y = X/G$ such that

$$\phi^* \mathcal{W}_Y = \mathcal{W}_1 \boxtimes \ldots \boxtimes \mathcal{W}_r.$$

Lemma (Decomposition lemma)

For every algebraic r-web $\mathcal{W}$, there exists a generically finite $\phi : X' \dashrightarrow X$ and r foliations $\mathcal{F}_1, \ldots, \mathcal{F}_r$ such that

$$\phi^* \mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_r \text{ (completely decomposable).}$$

Let $\mathcal{F} \subset \Omega^1_X$ be a foliation X (equivalently, $\sigma : X \dashrightarrow \mathbb{P}(T_X)$ or $F = \overline{\sigma(X)}$). A foliation can be broken in two parts:

(1) The **singular locus** $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ is the set of points x where $\Omega^1_X/\mathcal{F}$ is not locally free at x. Equivalently, $x \in \mathrm{Sing}(\mathcal{F})$ if and only if
   - σ is not regular at x,
   - $F_x = \mathbb{P}(T_{X,x})$.

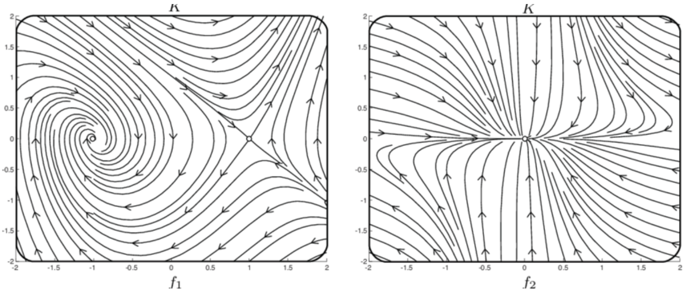
(2) $S = \mathrm{Sing}(\mathcal{F})$ is a finite subset of X and $\mathcal{F}$ defines a **regular foliation on** $U = X(\mathbb{C}) \setminus S$.

On $X(\mathbb{C}) \setminus S$, the foliation $\mathcal{F}$ defines an equivalence relation:

$x \sim_{\mathcal{F}} y$ if and only there exists an analytic curve $\gamma : \mathbb{D} \to X(\mathbb{C}) \setminus S$ joining x to y such that $\gamma'(t) \in F_{\gamma(t)}$ for all $t \in \mathbb{D}$.

Definition

The analytic leaves of $\mathcal{F}$ are the equivalence classes of this equivalence relation. They are analytic Riemann surfaces immersed in X^{an}.


K
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f_1
K
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
f_2

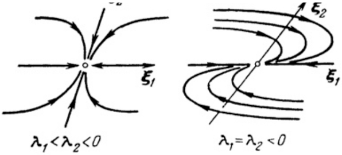


Fig. 126 An unstable focus.

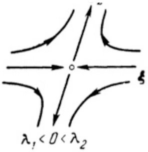


Fig. 127 A saddle point.

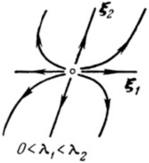


Fig. 128 An unstable node.

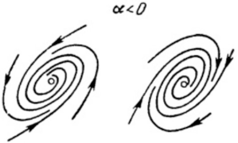


Fig. 129 Stable foci.

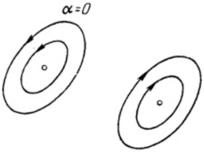


Fig. 130 Centers.

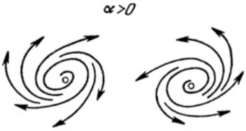


Fig. 131 Unstable foci.

Theorem (Jouanolou)

Let $\mathcal{F} \subset \Omega^1_X$ be a foliation X (equivalently, $\sigma : X \dashrightarrow \mathbb{P}(T_X)$ or $F = \overline{\sigma(X)}$). TFAE:

(i) *Infinitely many analytic leaves of $\mathcal{F}$ are algebraic curves on X.*

(ii) *All leaves of $\mathcal{F}$ are algebraic curves on X.*

(iii) *There exists an algebraic morphism $\phi : X \setminus Sing(\mathcal{F}) \to C$ such that the leaves of $\mathcal{F}$ are the connected components of the fibres of ϕ.*

- If these conditions are realized, we say that $\mathcal{F}$ is an **algebraically integrable foliation**.
- An algebraic r-web $\mathcal{W}$ is algebraically integrable if for some (equiv. any) generically finite cover such that

$$\phi^*\mathcal{W} = \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_r$$

  the foliations $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are algebraically integrable.
- A typical foliation $\mathcal{F}$ (resp. a typical web) is not algebraically integrable.

We now start moving from $\mathbf{ACF}_0$ to $\mathbf{DCF}_0$: our algebraic surface X **is endowed with a vector field** v i.e. a regular section of $p : T_X \to X$.
Locally, on a open set $U \subset X$ with étale coordinates (x, y), the vector field v can written as:

$$v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \text{ with } a, b \in \mathcal{O}_X(U).$$

We identify v with the derivation $\delta_v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ induced on $\mathcal{O}_X(U)$.

Lemma

There exists a unique extension of the derivation δ_v *to* $\mathcal{O}_X(U)[dx, dy]$ *denoted*

$$\mathcal{L}_v : \mathcal{O}_X(U)[dx, dy] \to \mathcal{O}_X(U)[dx, dy]$$

satisfying:

- $\mathcal{L}_v$ *is homogeneous of degree* 0.
- *For every* $f \in \mathcal{O}_X(U)$, $\mathcal{L}_v(df) = d(\delta_v(f))$.

This defines a global vector field w on T_X. In fact, $w = Tv$ where

$$T_v : T_X \to T(T_X)$$

obtained by functoriality.

Differential structure on $\mathbb{P}(T_X)$

- **Consequence:** if v is a vector field on X, we obtain a commutative diagram

$$\begin{array}{ccc} (TX \setminus \{\text{0-section}\}, Tv) & \longrightarrow & (\mathbb{P}(T_X), \mathbb{P}(v)) \\ \downarrow & \swarrow & \\ (X, v) & & \end{array}$$

  where $\phi : (X, v) \dashrightarrow (Y, w)$ means that ϕ is rational dominant morphism from X to Y satisfying $d\phi(v) = w$.

- Since Tv is "linear on the fibres", Tv descends to a vector field on $\mathbb{P}(T_X)$ (the global vector fields on $\mathbb{P}^1$ are the images in homogeneous coordinates of linear vector fields).

- It is a geometric variant of the classical result which asserts that if $y'' = ay' + by$ then y'/y satisfies the Ricatti equation:

$$z' = -z^2 + az + b.$$

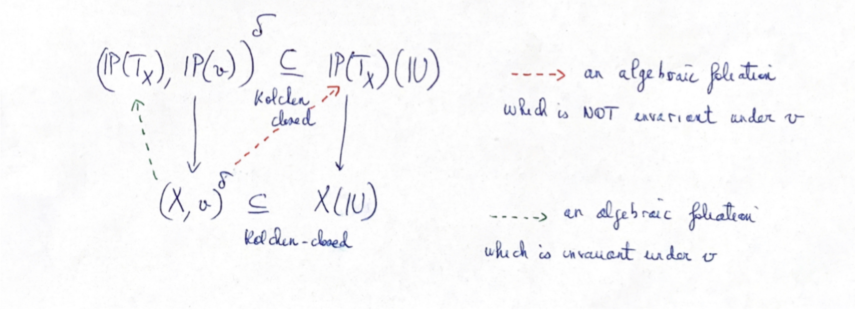
  This vector field extends to $\mathbb{P}^1$ (by computing $(1/z)'$ for example).

Proposition

Let v be a vector field on X and let $\mathcal{W} \subset Sym^k(\Omega^1_X)$ be an algebraic r-web on X and let W be the associated horizontal divisor. TFAE:

- *The horizontal divisor W (equivalently, all its irreducible components) is tangent to $\mathbb{P}(v)$.*
- *The web $\mathcal{W}$ is stable under $\mathcal{L}_v$ i.e. $\mathcal{L}_v(\mathcal{W}) \subset \mathcal{W}$.*

- When these conditions hold, we say that the web (or the foliation) $\mathcal{W}$ is **invariant under the vector field** v.
- Let $\mathcal{U} \models$ **DCF$_0$**.



Semi-minimality

Recall that a type is semi-minimal if it is almost internal to a minimal type.

Proposition ($T = \mathbf{DCF}_0$)

Let v be a vector field on X and denote by p the generic type of the differential equation (X, v).
*If there are **no algebraically integrable foliations on X invariant under the vector field** v then p is semi-minimal.*

- By geometric stability theory, if $p = tp(a/\mathbb{C})$ then there always exists $a_0 \in dcl(a, \mathbb{C}) \setminus \mathbb{C}$ such that $tp(a_0/\mathbb{C})$ is semi-minimal
- Since $\mathbb{C}\langle a \rangle$ has transcendence degree two, either p is semi-minimal or $\mathbb{C}\langle a_0 \rangle$ has transcendence degree one. (so assume the second case).
- There exists a curve C and a vector field w on C such that $tp(a_0/\mathbb{C})$ is intedefinable with the generic type of (C, w). The germ of a definable map sending a to a_0 defines

$$\phi : (X, v) \dashrightarrow (C, w)$$

- Denote by $\mathcal{F} = Ker(d\phi)$ the foliation tangent to the fibres of ϕ. By definition, this foliation is algebraically integrable.
- $d\phi(v) = w$ implies that the foliation $\mathcal{F}$ is invariant.

Theorem ($T = \mathbf{DCF}_0$)

Let v be a vector field on X and denote by p the generic type of the differential equation (X, v).
*Assume that p is **orthogonal to the constants**.*
*If X does not admit any **algebraically integrable 2-webs** nor **any algebraically integrable foliations** invariant under the vector field v then p is minimal and disintegrated.*

- Without the assumption that p is orthogonal to the constants, a counterexample is given by a global vector field (hence translation invariant) on a simple abelian variety A of dimension two.
- Are they essentially (up to a finite to finite correspondence) the only counterexamples?
- If w is a vector field on a curve C such that (C, w) is orthogonal to the constants. then set

$$X = S^2C = C \times C/\sim \text{ and } w = v \times v/\sim .$$

where $(x, y) \sim (y, x)$ admits an invariant algebraically integrable 2-web and no invariant algebraically integrable foliations. (Moosa-Pillay)

Proposition (Prototype statement, $T = \mathbf{CCM}$)

Let M be a compact complex surface. If M does not support any analytic web then the generic type of M is minimal and locally modular.

- M does not admit support analytic foliation, so in particular M does not support any "meromorphically integrable" analytic foliation.
- Like in $\mathbf{DCF}_0$, this implies that the generic type of M is semi-minimal.
- $\pi : T_M \to M$ do not have non-zero algebraic sections. So by GAGA Corollary, M is not almost internal to $\mathbb{P}^1$ so that M is orthogonal to $\mathbb{P}^1$.
- By Riemann existence theorem,

$$\text{semi minimal} + \text{orthogonal to } \mathbb{P}^1 \text{ in dimension two} \Rightarrow \text{minimal}$$

Remark: If we assume orthogonality to $\mathbb{P}^1$, we only need to look at analytic foliations and we don't need to consider analytic 2-webs.

A theorem in difference geometry

Theorem (C. Favre, J. V. Pereira, '14)

*Let $\sigma : X \dashrightarrow X$ be a dominant rational map with **positive entropy**.*
Assume that there exists an irreducible 2-web $\mathcal{W}$ invariant under σ and that σ does not admit any other invariant irreducible webs (in particular any foliation).
Then:

(1) *The 2-web $\mathcal{W}$ is algebraically integrable.*

(2) *There exists a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ such that*
- *$(\mathbb{P}^1, \phi)$ is not conjugated to a monomial, a Chebychev polynomial or a Lattes map.*
- *(X, σ) is semi conjugated (by a rational map) to*

$$S^2(\mathbb{P}^1, \phi) = (\mathbb{P}^1 \times \mathbb{P}^1, \phi \times \phi)/\sim$$

where $(x, y) \sim (y, x)$.

In fact, $S^2(\mathbb{P}^1) \simeq \mathbb{P}^2$.

Let X be an algebraic variety and v be a vector field on X.

- We have described the irreducible webs on X invariant by v as the algebraic solutions of a Ricatti equation given by the "projectivized Kolchin tangent bundle" over the differential field $(\mathbb{C}(X), \delta_v)$.
- We denote by G the connected component of the Galois group of this Ricatti equation (in other words, the Galois group of this Ricatti equation over $(\mathbb{C}(X), \delta_v)^{alg}$
- **A characteristic feature of differential algebra "over the constants"**: for every vector field v, the foliation $\mathcal{F}(v)$ tangent to v is tautologically invariant under v.

Consequence (Kovacic):

$$G \neq PSl_2(\mathbb{C}).$$

Proposition ($T = \mathbf{DCF}_0$)

Let v be a vector field on X such that $\mathcal{F}(v)$ is not algebraically integrable. Exactly one of the three following cases occur:

(1) $G \simeq Aff_2(\mathbb{C})$ *if and only if* $\mathcal{F}(v)$ *is the unique irreducible web invariant under* v.

(2) $G \simeq G_m(\mathbb{C})$ *if and only if there are exactly two irreducible webs invariant under* v. *One of them is* $\mathcal{F}(v)$ *and the other one* $\mathcal{W}$ *is:*

- (2a) *either a foliation,*
- (2b) *or a 2-web.*

(3) $G = 0$ *if and only if* v *admits at least three invariant irreducible webs if and only if* v *admits infinitely many.*

This follows from arguments of Kovacic on Ricatti equations.

Thank you for your attention!