LECTURE 1. MOTIVATION

The aim of this course is to be an introduction to some techniques based on model theory an differential algebra aimed to the of study transcendence questions about holomorphic functions satisfying algebraic differential equations.

Basic computational problem. Given holomorphic functions $f_1, \dots, f_k$ on some open domain U of the complex plane, each satisfying an algebraic differential equation

$$(E) : P(t, y, y', \dots, y^{(n)}) = 0$$

where P is a polynomial whose coefficients are rational or algebraic functions of t.

(Q1) *How to compute a system of generators of the ideal*

$$I(f_1, \dots, f_n) = \{Q \in \mathbb{C}(t)[X_1, \dots, X_n] \mid Q(t, f_1(t), \dots, f_k(t)) = 0 \text{ for all (non singular) values } t \in U\}?$$

This ideal $I(f_1, \dots, f_n)$ is the called the ideal of algebraic relations among $f_1, \dots, f_n$. Equally interesting questions also arise by replacing the ideal $I(f_1, \dots, f_n)$ by its differential analogue (equivalently, by considering the algebraic relations among the f_i and their successive derivatives). A simpler problem is simply to count the number of such algebraic relations:

(Q2) *How to compute*

$$\#\begin{cases} \textbf{independent} \text{ alg. relations} \\ \text{satisfied by } f_1, \dots, f_n \end{cases} = n - \mathrm{td}(f_1, \dots, f_n/\mathbb{C}(t))?$$

Here, $\mathrm{td}(f_1, \dots, f_n/\mathbb{C}(t))$ means the transcendence degree of the field $\mathbb{C}(t, f_1(t), \dots, f_n(t))$ generated by $f_1, \dots, f_n$ and $\mathbb{C}(t)$ in the field of meromorphic functions on U.

Prerequisites. We will assume some basic knowledge on the following subjects:

- differential Galois theory for linear differential equations: our standard reference will be the first chapter (p3-p36) of the book *Differential Galois theory* of Marius van der Put and Michael Singer.
- a basic course in model theory and ω-stable structures: our standard reference will be the first two chapters (p1-p43) of the book *Model Theory and Algebraic Geometry* edited by Elisabeth Bouscaren.
- some basic knowledge on algebraic geometry over the complex numbers: the first chapter of Robin Hartshorne's book *Algebraic geometry* will be our standard reference.

Picard-Vessiot linear differential Galois theory. In the case where the equations are *linear*, these type of problems can be handled using differential Galois theory. This approach is based on a construction

$$\mathrm{PV} : \text{(linear differential equations)} \longrightarrow \text{ differential ring over } \mathbb{C}(t) \text{ (or a differential field ext. of } \mathbb{C}(t))$$

which to a linear differential equation (L) associates the PV-ring (resp. the PV-extension) associated to (L). Recall that if (L) is given in a matrix form as

$$(L) : Y' = A \cdot Y \text{ where } A \in \mathrm{Mat}_n(\mathbb{C}(t))$$

the PV-extension K/k associated to (L) is characterized by the following properties:

- (L) admits a fundamental system of solutions $P \in GL_n(K)$ in K.
- K is generated (as a field extension over k) by the entries $p_{i,j}$ of the matrix P.
- K/k admits non new constants.

A fundamental lemma in the proof of the Galois correspondence is to show that these three properties characterize a unique differential field extension of k up to isomorphism and one sets

$$\mathrm{Gal}(L) = \mathrm{Aut}_\delta(K/k).$$

which measures the failure of the PV-extension to be well-defined up to a unique automorphism. Furthermore, this group can be identified with an algebraic group over the constants giving a Galois correspondence:

$$\text{(closed algebraic subgroups of } G) \leftrightsquigarrow \text{(differential subfields of } K/k).$$

Exercise (first-order linear differential equations)**.** *Consider a linear differential equation of order one*

$$(L) : y' = f(t) \cdot y \text{ where } f(t) \in \mathbb{C}(t)^{alg}$$

and set $k = \mathbb{C}(t, f(t))$.

(a) *Show that either* $\mathrm{Gal}(L/k) \simeq \mathbb{C}^*$ *or* $\mathrm{Gal}(L/k) \simeq \mathbb{Z}/n\mathbb{Z}$ *a,d that these two cases depends on whether* (L) *admits a no nonzero algebraic solution or not.*

(b) *Show that if* (L) *admits a algebraic solution then* (L) *admits an algebraic solution of the form*

$$y'(t) = \sqrt[n]{\frac{P(t, f(t))}{Q(t, f(t))}}$$

where P, Q *are polynomials.*

To illustrate the effectiveness of PV-theory to study the two basic computational problems for linear differential equations, we prove

Theorem A (Functional Lindemann-Weierstrass Theorem)**.** *Let* $f_1(t), \ldots, f_n(t) \in \mathbb{C}(t)^{alg}$ *be non constant algebraic functions defined on some complex domain* U*.*

$f_1(t), \ldots, f_n(t)$ *are* $\mathbb{Q}$*-lin ind. modulo* $\mathbb{C} \Rightarrow$ *their exponentials* $e^{f_1(t)}, \ldots, e^{f_n(t)}$ *are alg. ind. over* $\mathbb{C}(t)$

The proof will be in two steps. The first step will be to compute the Galois group of a linear differential equation of the form

$$y' = f'(t) \cdot y$$

when $f'(t) \in \mathbb{C}(t)$ is the derivative of an algebraic function $f(t)$. The second step is that a combination of Step 1 and the Galois correspondence gives the conclusion of the theorem.

Claim (Step 1)**.** *Let* $h(x) \in \mathbb{C}(x)^{alg}$ *be an algebraic function and*

$$(L) : y' = h'(x) \cdot y$$

the differential equation given by the derivative of $h(x)$*. Then* $\mathrm{Gal}(L) = G_m(\mathbb{C})$ *iff* $h(x) \notin \mathbb{C}$.

Proof. One direction is obvious. Assume $\mathrm{Gal}(L) \neq \mathbb{G}_m(\mathbb{C})$ and and that h is not a constant algebraic function for the sake a contradiction. The previous exercise implies that

$$(L) : y' = h' \cdot y$$

admits a nonzero algebraic solution $\phi \in \mathbb{C}(x)^{alg}$. To obtain an analytic realization, note that h as an algebraic function satisfies an (irreducible) algebraic equation of the form

$$y^n + a_{n-1}(x) \cdot y^{n-1} + \ldots + a_0(x) = 0$$

and we can realize h as an analytic function $h(x) \in \mathrm{Hol}(U)$ defined on a simply connected domain U avoiding the poles of the $a_i(x)$. Furthermore, up to restricting U even more, we may assume that

- $h(x) : U \to \mathbb{C}$ is a biholomorphism onto its image.
- the algebraic solution $\phi(x) : U \to \mathbb{C}$ on U is also an holomorphic function on U.

A direct computation shows that

$$\psi = \phi \circ h^{-1} : z \mapsto \phi(h^{-1}(z))$$

is a solution of the differential equation $y' = y$. Indeed, the chain rule gives:

$$\frac{d\psi}{dz} = \frac{d\psi}{dx}(h^{-1}(z)) \cdot \frac{dh^{-1}(z)}{dz} = h'(h^{-1}(z)) \cdot \phi(h^{-1}(z)) \cdot \frac{1}{h'(h^{-1}(z)}) = \psi.$$

To conclude, we will use the classical fact from analytic geometry/o-minimal geometry.

Fact (Analytic geometry)**.** *The class of algebraic functions is stable under composition and compositional inverse (whenever this operation make sense).*

It follows from this fact that ψ is an algebraic solution of $y' = y$ and hence must be equal to zero and hence so is ϕ which is our contradiction. □

Proof of the theorem. Let $f_1(x), \dots, f_n(x) \in \mathbb{C}(x)^{alg}$ be non constant, set $g_i(x) = e^{f_i(x)}$ and assume that $(*)$

$$\mathrm{td}(g_1(x), \dots, g_n(x)/\mathbb{C}(x)) < n$$

Note that the $g_i(x)$ are nonzero solutions of

$$(L_i) : y' = f_i'(x) \cdot y \text{ for } i = 1, \dots, n$$

and generate the PV-extensions associated to the (L_i) which by the previous claim all have Galois group $\mathbb{C}^*$. Now consider the composite

$$L = L_1 \cdots L_n = \mathbb{C}(t)^{alg}(g_1(t), \dots, g_n(t))/\mathbb{C}(t)^{alg}$$

the composite of all the PV-extensions associated to the (L_i). Since the composite of PV-extensions is a PV-extension, we can consider

$$G = \mathrm{Gal}(L/\mathbb{C}(t)^{alg}).$$

On the one hand, the action of G on L preserves each the L_i and define an algebraic group embedding:

$$i : \mathrm{Gal}(L/k) \to \mathrm{Gal}(L_1/k) \times \dots \times \mathrm{Gal}(L_n/k) = \mathbb{G}_m^n(\mathbb{C})$$

which is given by the formula

$$\sigma \mapsto (\sigma(f_1) \cdot f_1^{-1}, \dots, \sigma(f_n) \cdot f_n^{-1})$$

and on the other hand

$$\dim(\mathrm{Gal}(L/k)) = \mathrm{td}(L/k) < n$$

by assumption. It follows that the image of i the is a proper algebraic subgroup of $\mathbb{G}_m^n(\mathbb{C})$. The structure of the subgroups of the multiplicative torus then implies that any automorphism σ of G satisfies an equation of the form

$$\prod y_i^{e_i} = 1 \text{ where } y_i = \sigma(f_i) \cdot f_i^{-1}$$

Setting $g = \prod g_i^{e_i} \in L$ and reordering the factors, we obtain that $\sigma(g) = g$ for all $\sigma \in G$ and therefore by applying the Galois correspondence that $g \in \mathbb{C}(t)^{alg}$. Note that g is an algebraic solution of the linear differential equation

$$y' = (e_1 \cdot f_1 + \dots e_n \cdot f_n)' \cdot y$$

and hence by the applying Step 1 again, we conclude taht $e_1 \cdot f_1 + \dots + e_n \cdot f_n$ is a constant as required. □

Objectives of this course. The objective of this course is to present analogous techniques using model theoretic techniques to adapt the previous constructions to the case of algebraic differential equations.

$$\text{(linear differential equations)} \rightsquigarrow \text{(algebraic differential equations)}$$

Objective 1. Construct an *algebraic analogue of the construction* which to a linear differential equation associates its PV-ring and its PV-extension

$$\mathrm{Sol} : \text{(alg. differential equation)} \to \mathrm{Def}(\mathrm{DCF}_0)$$

which to an algebraic differential equation associates a definable set in the theory DCF_0 of differentially closed field of characteristic zero. The first objective of this course is a presentation of the category $\mathrm{Def}(\mathrm{DCF}_0)$ of definable sets of this theory. A fundamental result is:

Theorem B (Blum-Poizat)**.** *The theory $T = \mathrm{DCF}_0$ admits the elimination of quantifiers and imaginaries in the language of differential rings.*

The first part of the theorem — the elimination of quantifiers — ensures that every k-definable set of one variable can be presented as boolean combination of equations given by differential polynomials.

$$P \in k\{X\} = k[X, X', \dots, X^{(n)}, \dots]$$

and commutative algebra of the ring of differential polynomials will be our *first occupation in this course.* The second part says — the eliminations of imaginaries — says that one can take quotient in the category $\mathrm{Def}(\mathrm{DCF}_0)$: if D is a definable set and R is a definable equivalence relation on D, then R/D can be identified with an object of $\mathrm{Def}(\mathrm{DCF}_0)$.

Objective 2. *In this framework, write a proof of the Ax-Schanuel Theorem for the exponential function.*

Theorem C (Ax-Schanuel). *Let $f_1(t), \dots, f_n(t)$ be nonconstant holomorphic functions. Then*

$$\mathrm{td}(f_1(t), \dots, f_n(t), e^{f_1(t)}, \dots, e^{f_n(t)}/\mathbb{C}(t)) \geq n$$

provided $f_1(t), \dots, f_n(t)$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$.

This theorem of Ax is one of the most striking applications of the methods of differential algebra. It a functional analogue of a conjecture of number theory (Schanuel's conjecture) which predicts a similar behavior for the algebraic relations between complex numbers and their exponentials. In the special case where all the f_i are algebraic functions, we recover

$$\mathrm{td}(f_1(t), \dots, f_n(t), e^{f_1(t)}, \dots, e^{f_n(t)}/\mathbb{C}(t)) = \mathrm{td}(e^{f_1(t)}, \dots, e^{f_n(t)}/\mathbb{C}(t)) \geq n$$

which is Lindermann-Weierstrass Theorem. Finally, note that Ax Theorem admits a purely differential algebraic formulation replacing the analytic functions $f_1(t), \dots, f_n(t) \in \mathrm{Hol}(U)$ by (abstract) elements $f_1, \dots, f_n$ of a differential field extension $(K, \partial)/\mathbb{C}(t)$ *without new constants.*

no new constants. a differential field extension K/k has no new constants if K and k have the same (algebraically closed) field of constants.

Although, we can not directly make sense of precomposition by the exponential function in the differential setting, we have a close enough analogue. An element e of K is called an exponential of an element f of K (f is then called a logarithm of e) if they satisfy the differential relation

$$\partial(e)/e = \partial(f)$$

which is obviously satisfies by $f(t)$ and $e(t) = e^{f(t)}$ in the differential function field of $\mathrm{Hol}(U)$. Translating the previous theorem in this (weaker) setting, we obtain the following statement.

Theorem D (Ax Theorem, differential version). *Let $K/k(t)$ be a differential field extension without new constants, $f_1, \dots, f_n \in K$ and $e_1, \dots, e_n \in K$ exponentials of $f_1, \dots, f_n$. Then*

$$\mathrm{td}(f_1, \dots, f_n, g_1, \dots, g_n/k(t)) \geq n$$

provided $f_1, \dots, f_n$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$.

Objective 3. *Model theory of differential algebraic groups and of strongly minimal algebraic differential equations.*

More information about this objective will be added later on when we have advanced on the first two objectives.