LECTURE 7. THE FROBENIUS INTEGRABILITY THEOREM

The goal of this lecture is to present the following theorem.

Theorem 7.1. *Let M be a smooth affine complex variety and let $\omega \in \Omega^1(M/\mathbb{C})$ be a closed regular one-form. There exists a partition*

$$M = \bigsqcup_{\alpha \in A} L_\alpha \sqcup S$$

where S is a proper closed subvariety of M and the L_α are complex-analytic subspaces of $M \setminus S$ codimension one such that for every $\phi : Y \to M$, we have:

$$\phi^*\omega = 0 \in \Omega^1(Y/\mathbb{C}) \Leftrightarrow \phi(Y) \subset L_\alpha \text{ for some } \alpha \text{ or } \phi(Y) \subset S.$$

This theorem will be used in the proof of the Ax-Schanuel Theorem. We will describe this partition more precisely in the course of the proof. We start by recalling some basic properties of affine algebraic varieties. We refer to [Har77, Chapter 1] for more information.

7.1. **Regular functions.** Fix k be an algebraically closed field of characteristic 0 (e.g. $k = \mathbb{C}$). An *affine* algebraic variety M is a Zariski-closed set of k^n for some n. In other words, an affine variety is the solution set in k^n of a system

$$\begin{cases} P_1(x_1, \dots, x_n) & = 0 \\ & \vdots \\ P_s(x_1, \dots, x_n) & = 0 \end{cases}.$$

The Zariski-topology on k^n defines by restriction a *noetherian topology* on M. M is *irreducible* if it is irreducible for this noetherian topology. Assume that M is irreducible and set

$$k[M] = k[x_1, \dots, x_n]/\sqrt{P_1, \dots, P_s} \text{ and } k(M) = \mathrm{Frac}(k[M]).$$

$k[M]$ is called the coordinate ring of M and $k(M)$ the field of rational functions on M.

Definition 7.2. Let U be an open set of M. A function $f : U \to k$ is *regular at $x \in U$* if there exists a neighborhood V of x such that

$$f_{|V}(x_1, \dots, x_n) = \frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)} \tag{1}$$

where $P, Q \in k[x_1, \dots, x_n]$ and Q does not vanish on V. We say that f is a *regular function on U* if f is regular at every $x \in U$.

For every $U \subset M$, the space $\mathcal{O}_M(U)$ of regular functions on U is a k-algebra and $U \mapsto \mathcal{O}_X(U)$ is sheaf of rings on M. For every $p \in M$, we can also consider the ring of $\mathcal{O}_{M,p}$ of germs regular functions at $p \in M$. It is defined by:

$$\mathcal{O}_{M,p} = \lim_{p \in U} \mathcal{O}_X(U).$$

Definition 7.3. Consider the set $\mathcal{P}$ of pairs of the form (U, f) where U is open and f is a regular function on U. The relation defined by $(U, f) \sim (V, g)$ if and only if $f_{|U\cap V} = g_{|U\cap V}$ is an equivalence relation on $\mathcal{P}$. The quotient $\mathcal{P}/\sim$ is called the *field of rational fractions on X*.

Exercise 7.4. *Show that $\mathcal{P}/\sim$ is indeed a field which is isomorphic to $k(M)$ over k.*

Proposition 7.5. *Let M be an irreducible affine variety. We have that*

(a) $\mathcal{O}_M(M) \simeq k[M]$,

(b) *For every $p \in M$, the ring $\mathcal{O}_{M,p}$ is isomorphic to the localization of $k[M]$ at the maximal ideal m_p corresponding to p.*

Proof. (b). Recall that by the differential Nullstenllensatz, there is a one-to-one correspondence between maximal ideal m_p of $k[M]$ and points p in M (that is, minimal Zariski-closed subsets). The maximal ideal m_p is the set of $f \in k[M]$ satisfying $f(p) = 0$.

Now, by definition we have a morphism $f : k[x_1, \ldots, x_n] \to \mathcal{O}_{M,p}$ since every polynomial defines a regular function at p. The kernel is formed by the polynomials that vanishes on a Zariski neighborhood of p. Hence $\mathrm{Ker}(f) = \sqrt{(P_1, \ldots, P_s)}$ and f defines an injective morphism $k[M] \to \mathcal{O}_{M,p}$. Since for every $f \in k[M]$, $f(p) \neq 0$ implies that $1/f$ is regular at zero, this induces an injective morphism from the localization $k[M]_{m_p}$ to $\mathcal{O}_{M,p}$. By definition, every element in $\mathcal{O}_{M,p}$ can be represented as P/Q and hence it is surjective.

(a). Follows from (b) using the following fact: every integral domain is equal to the intersection of its localization around maximal ideals. □

Let $p \in M$ and consider $\mathcal{O}_{M,p} \to k(M) \xrightarrow{d} \Omega^1(k(M)/k)$. The universal property of differential forms implies that it factors by a $\mathcal{O}_{M,p}$-map

$$i_p : \Omega^1(\mathcal{O}_{M,p}/k) \to \Omega^1(k(M)/k).$$

Note that i_p is injective and identifies $\Omega^1(k(M)/k)$ to $\Omega^1(\mathcal{O}_{M,p}/k) \otimes_{\mathcal{O}_{M,p}} k(M)$.

Definition 7.6. A rational one-form $\omega \in \Omega^1(k(M)/k)$ is regular at $p \in M$ if it lies in the image of i_p. We denote by $\Omega^1(M/k)$ the space of regular one-forms on M.

As in Proposition 2.5 (c), we have that $\Omega^1(M/k) \simeq \Omega^1(k[M]/k)$: for every finitely generated module over an integral domain is the intersection of its localizations at maximal ideals.

Lemma 7.7. *Let $\phi : Y \to M$ be a morphism of affine varieties. We have a well-defined pullback morphism of $k[M]$-module*

$$\phi^* : \Omega^1(M/k) \to \Omega^1(Y/k).$$

Proof. If $\phi : Y \to M$ is a morphism of affine algebraic varieties then we have

$$D : \begin{cases} k[M] \to k[Y] \to \Omega^1(Y/k) \\ f \mapsto f \circ \phi \mapsto d(f \circ \phi) \end{cases}.$$

The $k[M]$-module on $\Omega^1(Y/k)$ is defined by $f \cdot \omega = (f \circ \phi) \cdot \omega$. The universal property of differential forms implies that it factors as a morphism of $k[M]$-modules $\phi^* : \Omega^1(M/k) \to \Omega^1(Y/k)$. □

Example 7.8. Take $M = k^2$ and define

$$\omega = dy/y - dx \in \Omega^1(k(x, y)/k)$$

Then ω is a regular at $p \in k^2$ if and only if p does not lie on the line given by $y = 0$. In particular, ω defines a regular one-form on $k \times k^*$ but not on k^2.

Exercise 7.9. *Define similarly a notion of regular point for a derivation $D \in \mathrm{Der}(k(M)/k)$. Show that $D \in \mathrm{Der}(k(M)/k)$ is regular at every point $p \in M$ if and only if it lies in $\mathrm{Der}(k[M]/k)$.*

7.2. **The tangent space of a smooth affine variety.** Let M be an irreducible affine subvariety of k^n defined by $P_1, \dots, P_s \in k[x]$. We have

$$\Omega^1(k[x]/k) \simeq k[x]dx_1 + \ldots + k[x]dx_n \subset k[x_1, \dots, x_n][dx_1, \dots, dx_n].$$

We identify $k[x_1, \dots, x_n][dx_1, \dots, dx_n]$ with the coordinate ring of $Tk^n \simeq k^{2n}$.

Definition 7.10. The *tangent space TM of M* is the Zariski-closed subset of k^{2n} given by

$$TM = V(P_1, \dots, P_s, dP_1 \dots, dP_s) \subset k^{2n} \simeq Tk^n.$$

By definition, the projection $\pi : Tk^n \to k^n$ defines by restriction $\pi : TM \to M$. The fiber over $p \in M$ is given by

$$TM_p = \{v \in k^n \mid dP_i(p)(v) = 0\}.$$

It is therefore a subvector space of k^n. M is *smooth* if its dimension does not depend on p. In that case, it coincides with the Krull dimension of M defined by its noetherian topology.

Definition 7.11. A vector field on a open set U of M is a regular section $v : U \to TU$ of the tangent bundle which is a regular map of algebraic varieties.

The space $\Xi(U)$ of vector fields on U is an $\mathcal{O}_M(U)$-module. Taking inductive limits over neighborhoods of p, we can also define the $\mathcal{O}_{M,p}$-module of germs of regular vector fields at p on M.

Fact 7.12 (Nakayama's lemma). *Let M be an irreducible affine variety. The following are equivalent:*

(i) *the variety M is smooth,*
(ii) *$\pi : TM \to M$ is a vector bundle, that is, every $p \in M$ admits a neighborhood V such that $\pi^{-1}(V) \simeq V \times k^n$ over V.*
(iii) *For every p, the $\mathcal{O}_{M,p}$-module $\Xi_{M,p}$ of germs of regular vector fields at p is free.*

Construction 7.13. Let U be an affine open set of M and $D \in \mathrm{Der}(\mathcal{O}_M(U)/k)$ a regular derivation on U. The coordinates functions $x_1, \dots, x_n$ on k^n define by restrictions regular functions $x_1| \, U, \dots, x_n \mid U$ on M and consider $v_D : M \to M \times k^n$ defined by

$$v_D(p) = (p, D(x_1| \, U)(p), \dots, D(x_n| \, U)(p)) \in M \times k^n$$

Proposition 7.14. *For every $D \in \mathrm{Der}(k(M)/k)$ regular on U, the map v_D takes values in $TM \subset M \times k^n$. Furthermore, if U is an affine open set then*

(i) *$D \mapsto v_D$ defines an isomorphism $\mathrm{Der}(\mathcal{O}_M(U)/k) \simeq \Xi(U)$ of $\mathcal{O}_X(U)$-modules*
(ii) *If $I \subset \mathcal{O}_X(U)$ is an ideal defining $Z = V(I)$ then I is a differential ideal for D if and only if $v_{D|Z} : Z \to TX$ takes values in TZ.*

Proof. To see that v_D takes values in TU, note that $P_i(x_1 \mid U, \dots, x_n \mid U) = 0$ for all i. Since D is a derivation, we get:

$$0 = D(P_i(x_1 \mid U, \dots, x_n \mid U)) = (dP_i(v_D)) \circ (x_1 \mid U, \dots, x_s \mid U)$$

This means that the function $dP_i(v_D)$ is identically equal to zero on U.

To prove (i), note that $V \mapsto v_D$ is injective since $x_1 \mid U, \dots, x_n \mid U$ generate the function field of $\mathcal{O}_M(U)$. It remains to show that is surjective. Consider an open set V of k^n such that $V \cap M = U$ so that the restriction morphism $\mathcal{O}(V) \to \mathcal{O}_M(U)$ is surjective and consider $v = (s_1, \dots, s_n)$ a vector field on U. Pick lifts $t_1, \dots, t_n$ of $s_1, \dots, s_n$ to $\mathcal{O}(V)$. By definition,

$$k[x_1, \dots, x_n] \subset \mathcal{O}(V) \subset k(x_1, \dots, x_n)$$

and hence there is a unique derivation $\partial_0 : \mathcal{O}(V) \to \mathcal{O}(V)$ such that $\partial_0(x_i) = t_i$ and which is trivial on k. To show that it descends to a derivation on $\mathcal{O}_M(U)$, it remains to see that the ideal $(P_1, \dots, P_s)$ is a differential ideal. indeed, if $P = P_i$ then

$$\partial_0(P)_{|U} = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}_{|U} \cdot s_i$$

is equal to zero in $\mathcal{O}_X(U)$ since v takes values in TM. Hence, $\partial(0)(P)$ belongs to the kernel of $O(V) \to \mathcal{O}_M(U)$ which is the ideal generated by $P_1, \dots, P_s$ as required.
(ii) is left as an exercise using the proof of (i). □

Corollary 7.15. *Assume that M is smooth. Let $\omega \in \Omega^1(k(M)/k)$ be a rational one-form and $x \in M$ be a point where ω is regular. Then ω defines a k-linear form $\omega(p) : T_pM \to k$ such that for every $X \in \Xi(U)$ defined on a neighborhood U of p*

$$\omega(X)(p) = \omega(p)(X(p)).$$

Proof. Since M is smooth, $\Xi_{M,p}$ is a free $\mathcal{O}_{M,p}$-module, every $v \in T_pM$ is of the form $X(p)$ for some local vector field X around p. Hence we can set $\omega(p) = \omega(X)(p)$. To see that it is well defined note that if $X(p) = 0$ then $X \in m_p \cdot \Xi_{M,p}$. Since ω is regular at p, it follows that $\omega(X)(p) = 0$. Hence $\omega(p)$ is well defined and satisfies the conclusion of the corollary. □

Remark 7.16. Let ω be a regular one-form on M and consider its pullback $\phi^*\omega$. Then

$$\phi^*\omega(p) = \omega(\phi(p)) \circ T\phi$$

where $T\phi : TY \to TM$ is obtained by functoriality of the tangent space construction.

7.3. **Proof of Theorem 7.1.** Let M be a smooth complex variety and $\omega \neq 0 \in \Omega^1(M/k)$ which is closed. Since ω is regular, it defines at every point p, a linear form

$$p \mapsto \omega(p) : T_pM \to k.$$

First step: Remove the zero locus. We claim that the subset

$$S = \{p \in M \mid \omega(p) = 0\}$$

is a proper Zariski-closed subset of M.

Proof. We have $S \neq M$ since $\omega \neq 0$. It remains to show that $U = M \setminus S$ is open. Take $p \in U$ and choose $v \in T_pM$ such that $TU \simeq U \times k^n$. Since $TM \to M$ is a vector bundle, we can find a neighborhood V of p and a vector field X such that

$$X : U \to TU$$

satisfies $X(p) = v$. Now the function

$$p \mapsto \omega(p)(X(p)) = \omega(D_X)(p)$$

is not identically zero since it does not vanish at p. Hence, its zero set Z is a Zariski-closed subset not containing p. □

Observation 2. Use closedness of ω. Set $U = M \setminus S$

$$H = \{(p, v) \in TU \mid \omega(p)(v) = 0\} \subset TU.$$

By construction H is a subvectorbundle of TU. We claim that for every $V \subset U$ and every $X, Y : V \to TV$ with values in H then

$$[X, Y] : V \to TV$$

also takes value in H.

Proof. Denote by $v \mapsto D_v$ the isomorphism $\mathrm{Der}(\mathcal{O}_M(V)) \simeq \Xi(V)$ given in the previous section. The Lie bracket of vector fields is given by

$$D_{[X,Y]} = [D_X, D_Y] = D_X \circ D_Y - D_Y \circ D_X.$$

Furthermore, we have that

$$\omega(D_v)(p) = \omega(p)(v(p))$$

by definition of $\omega(p)$. Using that ω is closed, we see that

$$0 = d\omega(D_X, D_Y) = D_X(\omega(D_Y)) - D_Y(\omega(D_X)) - \omega([D_X, D_Y]) = -\omega([D_X, D_Y]) = -\omega(D_[X, Y]).$$

It follows that $[X, Y]$ takes value in H. □

Observation 3. Apply complex differential geometry.

Fact 7.17 (Frobenius integrability theorem)**.** *Let M be a complex-analytic manifold and $H \subset TM$ a subvector bundle.*

Assume that the sections with values in H are stable under the Lie-bracket. Then for every $p \in M$, we can find a neighborhood U and a submersion $f : U \to \mathbb{C}$ such that for all p, we have

$$H(p) = T_p F_p \text{ where } F_p = f(f^{-1}(p)).$$

Definition 7.18. We say that two points $x, y \in M$ are H-*equivalent* if there exists a sequence

$$x_0 = x, x_1, \ldots, x_n = y$$

such that for every i, x_i and x_{i+1} lie in a analytic open set U_i equipped with an analytic submersion

$$f_i : U \to \mathbb{C}$$

tangent to the distribution $H(\omega)$ and such that $f_i(x_i) = f_i(x_{i+1})$.

By the Frobenius integrability theorem, H-equivalence is an equivalence relation on M. The equivalence classes are called the *leaves of* H. By construction, it satisfies the conclusion of Theorem 7.1

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.