LECTURE 3. DIFFERENTIALLY CLOSED FIELDS

We now apply the previous results to describe the model-theoretic properties of the theory DCF_0 of *differentially closed fields of characteristic zero*. This theory will be written in the language of differentials rings

$$\mathcal{L}_\partial = \{0, 1, +, \times, -, \partial\} = \mathcal{L}_{rings} \cup \{\partial\}$$

which is the language of rings expanded with a unary function symbol ∂. A $\mathcal{L}_\partial$-structure (K, ∂) is a model of DCF_0 if it satisfies the following (schemes of) axioms:

(A1) $K \models \mathrm{ACF}_0$ is an algebraically closed field of characteristic zero,

(A2) ∂ is additive and satisfies the Leibniz rule:

$$\partial(x + y) = \partial(x) + \partial(y) \text{ and } \partial(xy) = x\partial(y) + y\partial(x)$$

for all $x, y \in K$.

(A3) for every nonconstant differential polynomial $f, g \in K\{X\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$, there exists $x \in K$ such that

$$f(x) = 0 \wedge g(x) \neq 0.$$

Lemma 3.1. *The theory* DCF_0 *is consistent. Furthermore, any differential field* k *is contained in a model of* DCF_0.

Proof. Let k be a differential field and let $f, g \in k\{X\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$. Denote by f_1 an irreducible factor of f so that $\mathrm{ord}(f_1) = \mathrm{ord}(f)$ and consider $I(f_1)$ the prime differential ideal given by Theorem **??**. By construction of this ideal, $g \notin I(f_1)$ as g has lower order than f_1. It follows that the differential field

$$l = \mathrm{Frac}(k\{X\}/I(f_1))$$

extends k and contains an element a — the image of X — such that $f(a) = 0 \wedge g(a) \neq 0$.

- Iterating the process we produce a differential field extension $k_1 \mid k$ such that every system of equation and differential equations as above with coefficients in k has a solution in k_1.
- Iterating this new process, we obtain a chain of differential field extension

$$k \subset k_1 \subset k_2 \subset \ldots \subset k_n \subset \ldots$$

  such that every system of equation and differential equations as above with coefficients in k_i has a solution in k_{i+1}.

Clearly, the limit $k = \cup_{i \in \mathbb{N}} k_i$ is a differentially closed field containing k. □

3.1. Elimination of quantifiers.

Theorem 3.2 (Elimination of quantifiers)**.** *The theory* DCF_0 *admits the elimination of quantifiers in the language* $\mathcal{L}_\partial$ *of differential rings.*

Recall the following criterion for quantifier-elimination: Let T be a theory in a language $\mathcal{L}$. The theory T has QE in the language $\mathcal{L}$ if and only if

$(*)$ whenever $M, N \models T$ extend a common finitely generated substructure A, $\overline{a} \in A^n$, $m \in M$ and $\phi(x, \overline{y})$ a quantifier-free $\mathcal{L}$-formula (without parameters) such that

$$M \models \phi(m, \overline{a}) \Rightarrow N \models \exists x \phi(x, \overline{a})$$

Furthermore, up to replacing N by an elementary overstructure, we may assume that N is ω-saturated[1] in order to check $(*)$.

[1] This means that every countable set of $\mathcal{S} = \{\phi_i(x, \bar{l}) \mid i \in \mathbb{N}\}$ which is *finitely* satisfiable in N is satisfiable in N.

Proof. Consider $K, L \models \mathrm{DCF}_0$ containing a common finitely generated $\mathcal{L}_\partial$-substructure A and assume that L is ω-saturated. By definition of the language, A is a finitely generated substructure means that A is a finitely generated differential subring of K and L respectively and in particular is an integral domain. Denote by k the algebraic closure of the fraction field of A. Since *the derivation on A extends uniquely to k* and K, L are algebraically closed differential fields (by axioms (A1) and (A2)), the inclusions $i_K : A \to K$ (resp. $i_L : A \to L$) extends uniquely to inclusions

$$\overline{i_K} : k \to K \text{ (resp. } \overline{i_L} : k \to L)$$

Consider $m \in K$, $\overline{s} \in k$ and $\phi(x, \overline{y})$ quantifier-free such that $K \models \phi(m, \overline{s})$. To show that $L \models \exists \phi(x, \overline{s})$, a direct inspection of quantifier-free formulas[2] shows that it is enough to find $n \in L$ such that

$$k\langle m \rangle \simeq k\langle n \rangle$$

as differential fields over k sending m to n. If $m \in k$ then there is nothing to do. Otherwise, we distinguish according to the position of m with respect to k.

- Case 1. $m \in K$ satisfies a nontrivial differential equation over k that is

$$I(m/k) = \{f \in k\{X\} \mid f(m) = 0\}$$

is a nonzero ideal of $k\{X\}$.

By the first theorem of Ritt (Lecture 2), $I(m/k) = I(f)$ where f is a minimal nonzero polynomial in $I(m/k)$ with respect to $\ll$ and in particular is irreducible. The axiom (A3) of DCF_0 implies that the countable set of formulas

$$\{f(x) = 0 \wedge g(x) \neq 0 \mid g(x) \in k\{x\} \text{ with } \mathrm{ord}(g) < \mathrm{ord}(f)\}$$

is finitely satisfiable in L and hence by ω-saturation satisfiable in L. By hypothesis, $I(n/k)$ is a prime ideal containing f and no differential polynomial of lower order. Since f is irreducible, we have

$$I(n/k) = I(f) = I(m/k) \Rightarrow k\langle m \rangle \simeq k\langle n \rangle$$

as required.

- Case 2. $m \in K$ satisfies no nontrivial differential equation over k.

In that case, by ω-saturation of L, the countable set of formulas

$$\{f(x) \neq 0 \mid f(x) \in k\{X\}\}$$

is finitely satisfiable and hence satisfiable in L say by $n \in L$. By construction, we have

$$k\langle m \rangle \simeq k\langle X \rangle \simeq k\langle n \rangle.$$

as required. This completes the proof of the theorem. □

Corollary 3.3. *The theory* DCF_0 *is complete.*

Proof. Every differentially closed field contains $\mathbb{Q}$ equipped with the trivial derivation as a substucture. A theory with QE whose models share a common substructure is complete (exercise). □

3.2. **Geometric consequences.** *Fix for the rest of the section* $K \models \mathrm{DCF}_0$.

Definition 3.4. A *Kolchin-closed subset* Σ of K^n is a set of the form

$$\Sigma = \{\overline{x} \in K^n \mid f_1(x) = \ldots = f_n(x) = 0\}$$

where $f_1, \ldots, f_n \in K\{X_1, \ldots, X_n\}$ are differential polynomial of n variables. A Kolchin-closed set is called irreducible if it can not be written as the union

$$\Sigma = \Sigma_1 \cup \Sigma_2 \text{ with } \Sigma_1 \not\subset \Sigma_2 \text{ and } \Sigma_2 \not\subset \Sigma_1.$$

[2]The quantifier-free formulas with parameters from k are the boolean combination of formulas of the form $P(x, \overline{s}) = 0$ where $P \in k\{x\}$ is a differential polynomial

Corollary 3.5 (Differential Nullstenlensatz). *We have an inclusion reversing one-to-one correspondence*

$$\begin{array}{rcl} \{\ \textit{Kolchin-closed subset of}\, K^n\} & \leftrightarrows & \{\ \textit{radical differential ideals of } K\{\overline{X}\}\} \\ \Sigma & \to & I(\Sigma) = \{f \in K\{\overline{X}\} \mid f(x) = 0 \textit{ for all } x \in \Sigma\} \\ V(I) = \{\overline{x} \in K^n \mid f(\overline{x}) = 0 \textit{ for all } f \in I\} & \leftarrow & I \end{array}$$

Furthermore, the Kolchin-topology of K^n is a noetherian topology and irreducible Kolchin-closed subsets correspond to prime ideals.

Proof. Clearly, $I(\Sigma)$ is an ideal. It is radical and differential since for every $\overline{x} \in K^n$,

$$f^n(\overline{x}) = 0 \Rightarrow f(\overline{x}) = 0 \text{ and } f(\overline{x}) = 0 \Rightarrow \partial(f)(\overline{x}) = 0$$

as the evaluation is a morphism of differential rings. Conversely, $V(I)$ is a Kolchin-closed set since by the second theorem of Ritt (Lecture 2), $I = \{f_1, \ldots, f_n\}$ is finitely generated. Furthermore, we have

$$V(I(\Sigma)) = \Sigma \text{ and } I(V(\Sigma)) = I.$$

Indeed, the first equality follows from the second one as by definition any Kolchin-closed set Σ can be written $V(\{f_1, \ldots, f_p\})$ so that assuming the second equality, we get

$$V(I(\Sigma)) = V(I(V(\{f_1, \ldots f_p\}))) = V(\{f_1, \ldots, f_p\}) = \Sigma.$$

It is therefore enough to prove the second equality. To that end, note that $I \subset I(V(\Sigma))$ and consider $f \in K\{\overline{X}\} \setminus I$. Write

$$I = \bigcap_{j=1}^{n} I_j$$

where the I_j are prime differential ideals so that $f \notin I_j$ for some j. It follows that

$$L = \mathrm{Frac}(K\{\overline{X}\}/I_j) \subset \mathcal{U} \models \mathrm{DCF}_0$$

is a differential field. By construction, The image of $\overline{x}$ of $\overline{X}$ in $\mathcal{U}$ satisfies

$$\overline{x} \in \Sigma \wedge f(\overline{x}) \neq 0$$

so that $\mathcal{U} \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$ which is a sentence with parameters from K. It follows from Theorem 3.2 that modulo DCF_0, this formula is equivalent to a quantifier-free formula which is satisfied in $\mathcal{U}$ iff it is satisfied in K. It follows that

$$K \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$$

and hence that $f \notin I(V(\Sigma))$ as required. The second part of the statement is left as an exercise using the second theorem of Ritt. □

Corollary 3.6 (Description of types). *Let $k \subset K$ be a differential subfield. The function*

$$I : \begin{cases} S_n(k) & \to \mathrm{Spec}_\partial K\{X_1, \ldots X_n\} \\ p & \to I = \{f \in k\{X_1, \ldots, X_n \mid \text{“}f(x) = 0'' \in p\} \end{cases}$$

is a bijection where $S_n(k)$ denotes the model-theoretic space of types and $\mathrm{Spec}_\partial K\{X_1, \ldots X_n\}$ *is the set of differential prime ideals of $k\{X_1, \ldots, X_n\}$.*

Proof. We first need to show that I is well defined and that I is a prime ideal. Take $a \models p$. By enlarging K if necessary, we can find a realization $a = a_1, \ldots, a_n$ of p in a model of DCF. By construction of a, we have that

$$I = I(a) = \{f \in k\{X_1, \ldots, X_n\} \mid f(a) = 0\}$$

and the fact that I is a prime ideal follows easily from this presentation. It remains to show that I is injective and surjective. The second part is automatic. The first part follows directly from quantifier elimination: since every formula is equivalent to boolean combination of formulas of the form $f(x) = 0$, a type $p \in S_n(k)$ is determined by the function

$$f(x) \mapsto \chi_p : \begin{cases} 0 \text{ if “}f(x) = 0'' \in p \\ 1 \text{ otherwise} \end{cases}$$

which is the characteristic function of the subset I in $k\{X_1, \ldots, X_n\}$. Surjectivity follows from the differential Nullstellensatz as any partial type $\pi(x)$ (a consistent set of formulas) can be extended to a complete type. □

Theorem 3.7. *The theory* DCF_0 *is* ω*-stable (in the sense of model theory).*

Proof. A theory T in a countable language if for any infinite set of parameters A, we have that $\mid S_1(A) \mid \leq \mid A \mid$. Denote by k the differential field generated by A. Since A is infinite, we have $\mid k \mid = \mid A \mid$ and the restriction morphism

$$S_1(k) \to S_1(A)$$

is a bijection. Using the previous corollary together with the first theorem of Ritt (Lecture 2), we obtain that $\mid S_1(A) \mid = \mid S_1(k) \mid \leq \mid k \mid = \mid A \mid$ as required. □

3.3. **Elimination of imaginaries.** Recall that a complete theory T in a language of $\mathcal{L}$ admits the *elimination of imaginaries* if for every definable equivalence relation E on some definable set $D \subset M^n$ of some model $M \models T$, there exists a definable function $f : D \to M^s$ such that

$$xEy \Leftrightarrow f(x) = f(y).$$

One can then identify D/E with the definable set $f(D)$ and therefore under this condition take quotient without leaving the category of definable sets. To prove the elimination of imaginaries, we will use a rather indirect path. *Fix once for all* $K \models \mathrm{DCF}_0$ *an* ω *saturated and* ω*-homogeneous model.*

Definition 3.8. Let $\phi(x, a)$ be a formula (in the language of differential rings). A *differential field of definition* for $\phi(x, a)$ is a differential subfield $k \subset K$ such that there exists a formula $\psi(x, b)$ with parameters $b = b_1, \dots, b_n$ from k such that

$$\psi(x, b) \leftrightarrow \phi(x, a).$$

Similarly, if I is a differential ideal of $K\{X_1, \dots, X_n\}$, *a differential field of definition for* I is a differential subfield of K which contains a system of generators for I.

Lemma 3.9. *Let* $\phi(x, a)$ *be a formula and let* k *be a differential subfield of* K*.* k *is a field of definition of* $\phi(x, a)$ if and only if *for every* $\sigma \in \mathrm{Aut}_\partial(K)$*,*

$$\sigma \textit{ fixes } k \textit{ pointwise } \Rightarrow \sigma(D) = D \textit{ setwise}$$

where $D = \phi(K, a)$ *is the definable set defined by* $\phi(x, a)$*.*

Proof. The direct implication is obvious. To prove the converse, consider a differential subfield k such that for every $\sigma \in \mathrm{Aut}_\partial(K)$, if σ fixes k pointwise then $\sigma(D) = D$ setwise. Fix $b \in D$. By homogeneity of K, the assumption implies that any other realization of $p = \mathrm{tp}(b/k)$ also lies in D. So that

$$\mathrm{DCF}_0 \vdash p(x) \to \phi(x, a)$$

By compactness, we can find a formula $\psi_b(x) \in p(x)$ with parameters from k, such that $\mathrm{DCF}_0 \vdash \psi_b(x) \to \phi(x, a)$ and $K \models \psi_b(b)$. Since this is true for any $b \in D$, we conclude that

$$\mathrm{DCF}_0 \vdash \phi(x, a) \leftrightarrow \bigvee_{b \in D} \psi_b(x)$$

Using compactness again, we obtain that $\mathrm{DCF}_0 \vdash \phi(x, a) \leftrightarrow \bigvee_{i=1}^{n} \psi_{b_i}(x)$ which shows that k is a field of definition of $\phi(x, a)$. □

Proposition 3.10. *The following properties are equivalent:*

(i) $T = \mathrm{DCF}_0$ *admits the elimination of imaginaries,*
(ii) *every formula admits a smallest (finitely generated) differential field of definition,*
(iii) *every radical differential ideal of* $K\{X_1, \dots, X_n\}$ *admits a smallest (finitely generated) differential field of definition.*

Proof. (i) $\Rightarrow$ (ii). Let $\phi(x, a)$ be a formula with $a = a_1, \dots, a_n$. Consider the definable equivalence relation $E(y, z)$ on K^n defined by

$$E(y, z) \text{ iff } K \models \forall x(\phi(x, y) \leftrightarrow \phi(x, z))$$

and denote by $f_E : K^n \to K^m$ the function witnessing elimination of imaginaries. We first claim that the differential field k generated by $\alpha = f_E(a)$ is the smallest differential field k of definition of $\phi(x, a)$. Indeed, by construction

$$\sigma \text{ fixes } k \text{ pointwise } \text{ iff } \sigma(\alpha) = \alpha \text{ iff } K \models \phi(x, a) \leftrightarrow \phi(x, \sigma(a)) \text{ iff } \sigma(D) = D$$

and we conclude by the previous lemma that k is the smallest differential field of definition of $\phi(x,a)$.

(ii) $\Rightarrow$ (i) By Ritt-Raudenbush Theorem, every radical differential ideal I can be written as

$$I = \{f_1, \dots, f_s\}$$

where $f_1, \dots, f_s$ is a finite set of differential polynomials. Consider the formula

$$\phi(x,a) := \text{“}f_1(x) = 0 \wedge \ldots \wedge f_s(x) = 0''$$

where $a \in k$ is the tuple consisting of all the coefficients of the f_i. Clearly, by the differential Nullstenlensatz, a differential field of definition for $\phi(x,a)$ is a differential field of definition of I.

(iii) $\Rightarrow$ (i) Let $E(y,z)$ be a definable equivalence relation on some definable set D defined over k. For $a \in D$, denote by

$$[a]_E = \{x \in D \mid xEa\}$$

and by $\overline{[a]_E}$ its Kolchin-closure. We first claim that

Claim. *$\overline{[a]_E} = \overline{[b]_E}$ iff aEb.*

Proof of the claim. Indeed, clearly $aEb \Rightarrow [a]_E = [b]_E \Rightarrow \overline{[a]_E} = \overline{[b]_E}$. Conversely, if $\overline{[a]_E} = \overline{[b]_E}$ then $[a]_E$ contains a dense open Kolchin-subset U_a of $\overline{[a]_E}$ and so does $[b]_E$. Since any two dense Kolchin subset intersect, we have $U_a \cap U_b \neq \emptyset$ which implies (by transitivity) that aEb as required. □

Now fix $a \in D$, $p = \mathrm{tp}(a/k)$ and fix:

(i) α for a generator of the differential field of definition of $I(\overline{[a]_E})$,
(ii) $P_1(x,\alpha), \dots, P_s(x,\alpha)$ for generators of $I(\overline{[a]_E})$ with coefficients in $\mathbb{Q}\langle\alpha\rangle$

Note that since $[a]_E$ is $k\langle a\rangle$-definable so is $\overline{[a]_E}$ and therefore $\alpha \in k\langle a\rangle$ and there exits a k-definable function $f_a : D \to K^m$ such that $f_a(a) = \alpha$. By construction, if $\beta = f_a(b)$ then β satisfies the analogue of (i) and (ii) for $\overline{[b]_E}$.

Claim. *If $b_1, b_2 \models p$ then b_1Eb_2 iff $f_a(b_1) = f_a(b_2)$.*

Proof of the claim. Set $\beta_i = f_a(b_i)$. Clearly, if $\beta_1 = \beta_2$ then

$$I(\overline{[b_1]_E}) = \{P_1(x,\beta_1), \dots, P_s(x,\beta_1)\} = \{P_1(x,\beta_2), \dots, P_s(x,\beta_2)\} = I(\overline{[b_2]_E})$$

so that b_1Eb_2 by the previous claim. Conversely, assume that b_1Eb_2 so that $I(\overline{[b_1]_E}) = I(\overline{[b_2]_E})$ by the previous claim. Consider $\sigma \in \mathrm{Aut}_\delta(K/k)$ such that $\sigma(b_1) = b_2$ then by definition

$$\sigma(\overline{[b_1]_E}) = \overline{[b_2]_E} = \overline{[b_1]_E}$$

and it follows using that β_1 is fixed by every automorphism fixing $\overline{[b_1]_E}$ that

$$\beta_1 = \sigma(\beta_1) = \sigma(f_a(b_1)) = f_a(\sigma(b_1)) = f_a(b_2) = \beta_2$$

as required. □

Since this is true for any point $a \in D$, we can find by compactness a decomposition

$$D = D_1 \cup \ldots \cup D_r \text{ and } k\text{-definable functions } f_i : D_i \to K^{n_i}$$

such that for every $b, c \in D_i$, bEc iff $f_i(b) = f_i(c)$. We conclude the proof by building by induction on $i \leq r$ a k-definable function $g_i : D_1 \cup \ldots \cup D_i \to K^{m_i}$ with the same property: assume that g_i has been already build for $i < r$ and consider S_{i+1} the k-definable subset of D_{i+1} given by

$$S_{i+1} = \{x \in D_{i+1} \mid \exists z \in D_1 \cup \ldots \cup D_i \text{ such that } zEx\}$$

and consider $G \subset S_{i+1} \times K^{m_i}$ defined by

$$G := \{(x,y) \in S_{i+1} \times K^{m_i} \mid \exists z \in D_1 \cup \ldots \cup D_i \mid zEx \text{ and } g_i(z) = y\}$$

By the induction hypothesis, G is the graph of a k-definable function g. The function

$$g_{i+1}(x) = \begin{cases} g_i(x) \text{ if } x \in D_1 \cup \ldots \cup D_i \\ g(x) \text{ if } x \in S_{i+1} \\ f_{k+1}(x) \text{ otherwise.} \end{cases}$$

is an extension of g_i satisfying the required properties. This concludes the proof of the proposition. □

Theorem 3.11 (Field of definition of an ideal)**.** *Every ideal I of $K[X_1, \dots, X_n]$ admits a smallest field of definition.*

Proof. Denote by M a basis of monomials of $K[\overline{X}]/I$ as a K-vector space. Each monomial u of $K[\overline{X}]$ can be uniquely written as

$$u = \sum_{m \in M} a_{u,m} m + f_u$$

where $f_u \in I, a_{u,m} \in K$.

Claim. *The field*

$$k = \mathbb{Q}[a_{u,m} \mid u \text{ monomial of } K[\overline{X}], m \in M]$$

is the smallest field of definition of I.

- Step 1. *We show that k is a field of definition of I.*

For $f \in I$, we can write

$$f = \sum_{u \text{ mon. of } K[\overline{X}]} b_u u = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m\Big) + \sum_{m \in M} \Big(\sum_{u \text{ mon. of } K[\overline{X}]} b_u a_{u,m} \Big) \cdot m$$

Since by definition the left term lies in I and M is a K-basis of $K[\overline{X}]/I$, we conclude that all the coefficients of the right term must be zero and hence that

$$f = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m\Big).$$

It follows that I is generated by the $u - \sum_{m \in M} a_{u,m} m \in k[\overline{X}]$ where u ranges over all monomials of $K[\overline{X}]$ so that k is indeed a field of definition for I.

- Step 2. *Consider l another field of definition of I. We show that $k \subset l$.*

Note that every automorphism of K extends to an automorphism of $K[X_1, \dots, X_n]$ by setting:

$$\sigma\Big(\sum_{m \in mon.k[X]} f_m \cdot m\Big) = \sum_{m \in mon.k[X]} \sigma(f_m) \cdot m$$

Since l is a field of definition of I, for every $\sigma \in \mathrm{Aut}(K/l)$, we have $\sigma(I) = I$. It follows that for every monomial u, we have

$$u = \sigma(u) = \sum_{m \in M} \sigma(a_{u,m}) \cdot m + \sigma(f_u)$$

By uniqueness of the decomposition, it follows that $\sigma(a_{u,m}) = a_{u,m}$ for every $\sigma \in \mathrm{Aut}(K/l)$ and every u, m. We have therefore shown that k is a subset of l. ☐

Theorem 3.12 (Elimination of imaginaries)**.** *The theory* DCF_0 *eliminates imaginaries in the language $\mathcal{L}_\partial$ of differential rings.*

Proof. It is enough to show that every radical differential ideal I admits a smallest differential field of definition using Proposition 3.10. By the Ritt-Raudenbush Theorem, we can find a finite set of differential polynomial such that

$$I = \{f_1, \dots, f_n\}$$

Consider N large enough so that $f_1, \dots, f_n \in K[X, X', \dots, X^{(N)}]$ and set J for the ideal they generate. By Theorem 3.11, J has a smallest field of definition $k \subset K$. It is now easy to see that the differential field $\tilde{k}$ generated by k is the smallest differential field of definition of I. ☐

Example 3.13. Let $K \models \mathrm{DCF}_0$ and denote by $\mathcal{C}$ the field of constants of K.

- The imaginary K^*/C^* is eliminated by the function

$$\partial log : y \mapsto \partial(y)/y$$

- Consider the action of the affine group $Aff_2(C)$ on the affine line K. The imaginary $K/Aff_2(C)$ is eliminated by the affine distorsion:

$$y \mapsto \partial^2(y)/\partial(y)$$

- Consider the action of $PSL_2(C)$ on the projective line $\mathbb{P}^1(K)$. The imaginary $\mathbb{P}^1(K)/PSL_2(C)$ is eliminated by the Schwarzian derivative

$$y \mapsto (y''/y')' - 1/2(y''/y')^2$$

3.4. **References.** The concept of differentially closed field was introduced by Abraham Robinson [Rob59] as a differential analogue of the concept of algebraically closed field. The presentation of the theory DCF_0 based on the schemes of axioms (A1) to (A3) and the fact that differentially closed fields enjoy the elimination of quantifiers in the language of differential rings is due to Lenore Blum in her PhD thesis [Blu69]. Finally, the fact that differentially closed fields also enjoy the elimination of imaginaries in this language is due to Poizat [Poi83]. This lecture follows the presentation of these results in the lectures notes of Dave Marker in [MMP96] and the effective procedure for eliminating imaginaries in the theory of [Sca18].

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