On the problem of integration in finite terms and exponentially algebraic functions

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DART XII - Kassel

1. Joint work with Jonathan Kirby (University of East Anglia)

Outline

The purpose of my talk is to present the relationships between two classical theorems from differential algebra.

- **Theorem** (Liouville, 1835). Let $f(z)$ be an algebraic function. If the primitive $\int f(z)dz$ is an elementary function then it can be written as :

$$\int f(z)dz = \underbrace{R_0(z, f(z))}_{\text{rational function}} + \overbrace{\sum_{i=1}^{n} c_i \cdot \ln(R_i(z, f(z)))}^{\text{weighted sum of logs}}$$

  where $c_1, \ldots, c_n$ are complex numbers and $R_0, \ldots, R_n \in \mathbb{C}(X, Y)$.

- **Theorem** (Ax, 1971). Let $f_1(z), \ldots, f_n(z)$ be holomorphic functions. Then

$$\# \left\{ \begin{array}{l} \text{ind alg. relations which hold} \\ \text{of } f_1(z), \ldots, f_n(z) \\ \text{and } e^{f_1(z)}, \ldots, e^{f_n(z)} \end{array} \right. \leq n + \# \left\{ \begin{array}{l} \text{ind } \mathbb{Q}\text{-linear relations} \\ \text{among } f_1(z), \ldots, f_n(z) \\ \text{mod } \mathbb{C} \end{array} \right. .$$

- **In the background.** This is part of a wider project which aims to obtain a model-theoretic understanding (in the sense of mathematical logic) of the relationships between exponential algebra and differential algebra.
- Joint work with Jonathan Kirby (University of East Anglia).

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Liouville (1830-1840)

Motivation. In certain special cases, we can compute "explicitly" a primitive

$$\int \ln(z)dz = z \cdot \ln(z) - z \text{ or } \int \frac{dz}{\sqrt{1-z^2}} = \arccos(z)$$

but in general we can't. For example,

$$\int e^{-z^2} dz = ?? \text{ or } \int \frac{dz}{\sqrt{z^3+az+b}} = ??$$

- To formalize this idea, we follow the approach of Liouville based on the notion of elementary function.

$$\begin{cases} \text{solvability by elementary} \\ \text{functions for primitives} \end{cases} \leftrightarrows \begin{cases} \text{solvability by radicals} \\ \text{for algebraic equations.} \end{cases}$$

- Liouville's methods are not (differential) Galois-theoretic. Even assuming some knowledge of Picard-Vessiot differential Galois theory,

$$y' = \frac{1}{\sqrt{1-z^2}} \text{ and } y' = \frac{1}{\sqrt{z^3+az+b}} \text{ both have Galois group } \mathbb{G}_a.$$

while the first one defines an elementary function ($f(z) = \mathrm{arcos}(z)$) and not the second one ($f(z)$ is an elliptic integral).

Mémoire sur la classification des transcendantes (Liouville, JMPA, 1837)

Mais ici l'on voit naître une question semblable à celle qui s'est présentée tout-à-l'heure, quand nous parlions des fonctions algébriques. En effet, on a cru d'abord que toutes les équations algébriques se résoudraient à l'aide de radicaux; et, d'après cette idée, on a cherché long-temps à en obtenir les racines sous la forme indiquée. Les efforts réitérés des plus grands géomètres n'ayant conduit à aucun résultat général, on a été porté ensuite à soupçonner que le problème proposé était impossible, au moins pour l'équation complète du cinquième degré et des degrés supérieurs, et on est parvenu à établir en toute rigueur cette impossibilité: semblablement, quand il s'agit d'équations transcendantes, il est naturel de chercher d'abord à les résoudre, en exprimant les inconnues par des fonctions finies

explicites des coefficients, et comme on ne peut pas y réussir dans la plupart des cas, il faut en second lieu prouver que les valeurs des inconnues ne sont pas exprimables par cette sorte de fonctions. Dès-lors on aura épuisé complétement la question dans le sens où elle était proposée; car tout ce que peut faire une méthode, c'est de conduire à la solution, quand cette solution est possible, ou d'en prouver sans équivoque l'impossibilité.

Dans le mémoire que j'ai l'honneur de soumettre au jugement de l'Académie, je suis bien loin d'avoir envisagé la chose sous un point de vue aussi étendu. Je me suis contenté de traiter certaines équations particulières, et par un procédé direct et uniforme, qu'il serait facile de présenter d'une manière abstraite et générale, je suis parvenu soit à les résoudre, soit à démontrer l'impossibilité de leurs racines en fonction finie explicite des coefficients.

The class of elementary functions

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function (of one complex variable).

Definition

The function $f(z)$ is an elementary function (in the sense of Liouville) if — after possibly restricting its domain of definition — it can be constructed from the variable z using finitely many times the following operations :

(a) arithmetic operations $(+, \times, -, /)$ and multiplication with complex numbers,

(b) precomposition with the exponential function $\exp$ and (branches of) the logarithm $\ln$,

(c) extraction of roots of algebraic equations.

- **Examples.**
  - $R(z) = P(z)/Q(z), R(z, \sqrt{z})$ and more generally any algebraic function,
  - all trigonometric functions and their inverses : $\cos(z), \arccos(z)$...
  - arbitrary complicated compositions of such functions such as

$$f(z) = \frac{e^{\cos(z)}}{\sqrt{1+z^2}}(1 + \ln(1+z))...$$

- **The problem of integration in finite terms.** Find an algorithm which given an elementary function
  (1) decide whether its primitive is elementary,
  (2) if it is, compute an explicit construction of it using (a),(b),(c).

Liouville's Theorem

- **Theorem** (Liouville, relative version). Let $f(z)$ be an elementary function. If $\int f(z)dz$ is also elementary then it is of the form

$$\int f(z)dz = \underbrace{R_0(z, f(z), f'(z), \ldots, f^{(n)}(z))}_{\text{rational function}} + \overbrace{\sum_{i=1}^{m} c_i \cdot \ln(R_i(z, f(z), \ldots, f^{(n)}(z))}^{\text{weighted sum of logs}}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ and $R_0, \ldots, R_m \in \mathbb{C}(X_0, \ldots, X_n)$.

- In particular for $f(z) = e^{-z^2}$, $f'(z) = -2zf(z)$ and Liouville reaches :

$$\int e^{-z^2} dz = R_0(z, e^{-z^2}) + \sum_{i=1}^{m} c_i \cdot \ln(R_i(z, e^{-z^2}))$$

and shows that it does not admit any solution.

- For algebraic functions $f(z)$...

Theorem (Risch,1969 - Davenport, 1981). The problem of integration in finite terms for algebraic functions is decidable.

- Additionally, Risch shows the decidability of various properties of elementary functions such as checking the equality between two elementary functions.

In general, Risch's algorithms are not efficient.

Possible improvements on Liouville's theorem

(1) Replace functions of one variable by functions of several variables (Rosenlicht, 1960).

(2) **Adding a Galois-theoretic component.** instead of taking the primitive, consider elementary solutions of arbitrary linear differential equations.

Theorem (Davenport-Singer, 1986). Consider an inhomogeneous linear differential equation

$$(L) : y^{(s)} + a_{s-1}(z) \cdot y^{(s-1)} + \ldots + a_{-1}(z) \cdot y = b(z)$$

where $a_i(z), b(z) \in \mathbb{C}(z)^{alg}$. If (L) has an elementary solution, then (L) has a solution of the form

$$f(z) = P(z, \ln(R_1(z)), \ldots, \ln(R_m(z)))$$

where $P(z, -) \in \mathbb{C}(z)^{alg}[X_1, \ldots, X_m]$ is a polynomial of degree $\leq s$, $R_i(z) \in \mathbb{C}(z)^{alg}$.

(3) **Enlarging Liouville's class of elementary functions.** instead of Liouville's class of elementary functions, consider integrability properties with respect to larger classes of functions.

Theorem (Pila-Tsimerman, 2022). The problem of integration of an algebraic function by elementary functions and elliptic integrals is decidable.

- In this talk, we consider the enlargement obtained by closing Liouville's class of elementary functions under the implicit function theorem.

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The Ax-Schanuel Theorem

- **Schanuel's conjecture** (1960s). Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be complex numbers.

$$\mathrm{adim}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n}/\mathbb{Q}) \geq \mathrm{ldim}_{\mathbb{Q}}(\alpha_1, \dots, \alpha_n).$$

  - Famous examples : e and π are transcendental numbers.
  - Famous open problem : e and π are algebraically independent.

- The Ax-Schanuel Theorem is a functional analogue of Schanuel's conjecture.

Theorem (Ax, 1971). Let $f_1(z), \dots, f_n(z)$ be holomorphic functions. Then

$$\mathrm{adim}(f_1(z), \dots, f_n(z), , e^{f_1(z)}, \dots, e^{f_n(z)}/\mathbb{C}(z)) \geq \mathrm{ldim}_{\mathbb{Q}}(f_1(z), \dots, f_n(z) \text{ mod } \mathbb{C})$$

or equivalently :

$$\#\begin{cases} \text{ind alg. relations which hold} \\ \text{of } f_1(z), \dots, f_n(z) \\ \text{and } e^{f_1(z)}, \dots, e^{f_n(z)} \end{cases} \leq n + \#\begin{cases} \text{ind } \mathbb{Q}\text{-linear relations} \\ \text{among } f_1(z), \dots, f_n(z) \\ \text{mod } \mathbb{C} \end{cases}.$$

- **Slogan**. The Ax-Schanuel Theorem describes a "universal" inequality for (tuples of) holomorphic functions. Liouville's theorem describes the behavior of integration for functions lying on the equality case of this universal inequality.

The class of exponentially algebraic functions

Definition (Zilber, Kirby, around 2000-2010)

An holomorphic function $f(z)$ is exponentially algebraic if — after possibly restricting its domain of definition — it satisfies an algebraic equation of the form

$$y^n + a_{n-1}(z) \cdot y^{n-1} + \ldots, a_1(z) \cdot y + a_0(z) = 0$$

where the coefficients

$$a_i(z) = H_i(z, f_1(z), \ldots, f_n(z), e^{f_1(z)}, \ldots, e^{f_n(z)})$$

depends rationally on a tuple of nonconstant holomorphic functions $f_1(z), \ldots, f_n(z)$ and their exponentials satisfying the equality case in Ax Theorem.

Example. the function $f(z) = \frac{e^{cos(z)}}{\sqrt{1+z^2}}(1 + \ln(1+z))$ is exponentially algebraic Indeed, set

$$\begin{cases} f_1(t) = iz \\ e^{f_1(z)} = e^{iz} \end{cases}, \begin{cases} f_2(z) = e^{iz} \\ e^{f_2(z)} = e^{e^{iz}} \end{cases}, \begin{cases} f_3(t) = e^{-iz} \\ e^{f_2(t)} = e^{e^{-iz}} \end{cases}, \begin{cases} f_4(z) = \ln(1+z) \\ e^{f_4(t)} = 1 + z \end{cases}$$

so that $f(z) \in \mathbb{C}(z, e^{e^{iz}}, e^{e^{-iz}}, \ln(1+z))^{alg}$ while

$$\mathrm{td}(f_1(z), \ldots, f_4(z), e^{f_1(z)}, \ldots, e^{f_4(z)}/\mathbb{C}(z)) \leq 8 - 4 = \mathrm{ldim}_{\mathbb{Q}}(f_1(z), \ldots, f_4(z) \bmod \mathbb{C})$$

More examples

Obvious examples.

- Any algebraic function is exponentially algebraic and every exponentially algebraic function is differentially algebraic.
- Every elementary function is exponentially algebraic.

Compositional inverses of elementary functions.

- Consider the elementary function $f(z) = z \cdot e^z$.

The Lambert function $L(w)$ — Lambert (1780) — is a local analytic inverse of $f(z) = w$. Clearly, the relation $L(w) \cdot e^{L(w)} = w$ implies that

$$\mathrm{adim}(L(w), e^{L(w)}/\mathbb{C}(w)) \leq 1 = \mathrm{ldim}_{\mathbb{Q}}(L(w) \text{ mod } \mathbb{C})$$

so that $L(w)$ is exponentially algebraic. But it is well-known that $L(w)$ is not an elementary function of w (Liouville, 1837).

- Similarly for the Kepler function $K(w)$ which is a compositional inverse of

$$g(z) = z - e \cdot \sin(z).$$

Solutions of elementary functional equations. More generally, if $F(w, z)$ is an elementary function of two variables then any solution $w \mapsto z(w)$ of the functional equation

$$F(w, z) = 0 \text{ and } \frac{\partial F}{\partial z} \neq 0$$

is exponentially algebraic and in general a non-elementary function of w.

The structure theorem for exponentially algebraic functions

Let $f(z)$ be an holomorphic function (of one complex variable).

Theorem (Wilkie 2008, Kirby 2010)

After possibly restricting the domain of definition of the function $f(z)$, the following conditions become equivalent :

(i) *(Ax-Schanuel) the function $f(z)$ is exponentially algebraic,*

(ii) *(implicit definition) the function $f(z)$ can be constructed in finitely many steps from Liouville's class of elementary functions (with several variables) using the following operations*

- (α) *composition (with a possibly non fixed number of variables),*
- (β) *partial holomorphic derivatives,*
- (γ) *extraction of implicit functions under the analytic implicit function theorem.*

In short, we say the function $f(z)$ is an implicitly elementary function.

(iii) *(o-minimal characterization) the function $f(z)$ is definable in the o-minimal structure*

$$(\mathbb{R}, \exp_{|[-1,1]}, \sin_{|[-1,1]}).$$

obtained by expanding semi-algebraic geometry by the restricted exponential and the restricted sine functions only.

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A model-theoretic version of Liouville's Theorem

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function.

Theorem (J., Kirby)

Assume that both :

(a) *the function $f(z)$ is exponentially algebraic (equiv. implicitly elementary),*

(b) *the function $f(z)$ belongs to some Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$.*

Then :

- *(Galois-theoretic information). The Galois group of the smallest Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$ containing $f(z)$ is isomorphic to $\mathbb{G}_a^k \times \mathbb{G}_m^l$.*

The Galois-theoretic information allows effective computations.

- **Corollary** (Liouville). The three functions

$$\mathrm{erf}(z) = \int e^{-z^2} dz, \ \mathrm{sn}(z) = \int \frac{dz}{\sqrt{z^3 + az + b}} \text{ and } \mathrm{Ai}(z) = \int_0^\infty \cos(u^3/3 + uz) du$$

are not implicitly elementary functions.

Proof

- The three functions satisfy respectively the linear differential equations

$$\underbrace{y'' - 2z \cdot y' = 0}_{\mathrm{Aff}_2(\mathbb{C})}, \ \underbrace{y' = \frac{1}{\sqrt{z^3 + az + b}}}_{\mathrm{G}_a(\mathbb{C})} \text{ and } \underbrace{y'' - z \cdot y = 0}_{\mathrm{SL}_2(\mathbb{C})}.$$

- In each case, the Galois group acts transitively on the nonzero solutions of the equation. This gives the minimality requirement for the PV-extensions associated to each equation.
- For $\mathrm{erf}(z)$ and $\mathrm{Ai}(z)$, the Galois-theoretic information is conclusive. The compositional inverse of $\mathrm{sn}(z)$ satisfies the elliptic equation

$$\underbrace{(y')^2 = y^3 + ay + b}_{\text{an elliptic curve } E(\mathbb{C})}$$

- A conclusive test for $\mathrm{sn}(z)$ is given by a slight improvement of the previous theorem.
  (b) the function $f(z)$ belongs to some strongly normal extension (in the sense of Kolchin) of $\mathbb{C}(z)^{alg}$.

The Galois-theoretic information allows effective computations.

- **Corollary** (Liouville). The three functions

$$\mathrm{erf}(z) = \int e^{-z^2} dz,\ \mathrm{sn}(z) = \int \frac{dz}{\sqrt{z^3 + az + b}} \text{ and } \mathrm{Ai}(z) = \int_0^\infty \cos(u^3/3 + uz) du$$

  are not implicitly elementary functions.

- With the same proof but using a relative version of the previous theorem

$$\mathbb{C}(z)^{alg} \dashrightarrow \text{ any self-sufficient algebraically closed field differential field } k$$

  we obtain that the three previous examples are independent in the following sense.

- **Corollary**. For any choice of (implicitly) elementary functions ψ, ϕ, χ, the three functions

$$\psi \circ \mathrm{erf}(z), \phi \circ \mathrm{sn}(z) \text{ and } \chi \circ \mathrm{Ai}(z)$$

  — when the domains of definition are matching — are algebraically independent.

- **Main improvement over past results**(Magid, 1990s). the "good" Galois-properties of elementary functions are preserved after application of the implicit function theorem.

A natural question. Does the classical characterization of solvability by Liouvillian functions goes through for solvability by implicitly Liouvillian functions ?

A model-theoretic version of Liouville's Theorem

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function.

Theorem (J., Kirby)

Assume that both :

(a) *the function $f(z)$ is exponentially algebraic (equiv. implicitly elementary),*

(b) *the function $f(z)$ belongs to some Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$.*

Then :

- *(Galois-theoretic information). The Galois group of the smallest Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$ containing $f(z)$ is isomorphic to $\mathbb{G}_a^k \times \mathbb{G}_m^l$.*
- *(explicit definition). The function $f(z)$ is in fact an elementary function : a construction of $f(z)$ does not involve the implicit function theorem.*

Furthermore, an improved formulation includes a relative version of the theorem and the case of strongly normal extensions in the sense of Kolchin.

Transferring results from elementary functions...

(1) **Corollary** (Abel's version of Liouville's Theorem). Let $f(z)$ be an algebraic function. If the primitive $\int f(z)dz$ is an implicitly elementary function then it can be written as :

$$\int f(z)dz = R_0(z, f(z)) + \sum_{i=1}^{n} c_i \cdot \ln(R_i(z, f(z)))$$

where $c_1, \ldots, c_n$ are complex numbers and $R_0, \ldots, R_n \in \mathbb{C}(X, Y)$.

- In a recent article (2022), Singer explains that already Abel was aware of this corollary but that we have no trace of Abel's precise statement.

Modern proofs of this corollary were already obtained by Ritt and Risch.

(2) Furthermore, Risch's algorithm decide whether $\int f(z)dz$ is implicitly elementary or not.

(3) **Corollary** (implicit Davenport-Singer). Consider an inhomogeneous linear differential equation

$$(L) : y^{(s)} + a_{s-1}(z) \cdot y^{(s-1)} + \ldots + a_{-1}(z) \cdot y = b(z)$$

where $a_i(z), b(z) \in \mathbb{C}(z)^{alg}$. If (L) has an implicitly elementary solution, then (L) has a solution of the form

$$y(z) = P(z, \ln(R_1(z)), \ldots, \ln(R_m(z)))$$

where $P(z, -) \in \mathbb{C}(z)^{alg}[X_1, \ldots, X_m]$ is a polynomial of degree $\leq s$.

A model-theoretic version of Liouville's Theorem

Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function.

Theorem (J., Kirby)

Assume that both :

(a) *the function $f(z)$ is exponentially algebraic,*

(b) *the function $f(z)$ belongs to some Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$.*

Then :

- *(Galois-theoretic information). The Galois group of the smallest Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$ containing $f(z)$ is isomorphic to $\mathbb{G}_a^k \times \mathbb{G}_m^l$.*
- *(explicit definition). The function $f(z)$ is in fact an elementary function : a construction of $f(z)$ does not involve the implicit function theorem.*
- *(Liouville's theorem). there are finitely many algebraic functions $u_i(z), v_j(z), w_k(z) \in \mathbb{C}(z)^{alg}$ such that we can write*

$$f(z) = H(\ln(u_i(z)), \exp(v_j(z)), (w_k(z))^{\lambda_k})$$

where $H \in \mathbb{C}(X_0, \ldots, X_N)^{alg}$ is an algebraic function and the λ_k are complex numbers.

Furthermore, an improved formulation includes a relative version of the theorem, the case of strongly normal extensions in the sense of Kolchin and the case of holomorphic functions of several variables.

Comparison with the classical results

- Liouville : if $f(z)$ is an algebraic function and $\int f(z)dz$ is elementary then :

$$\int f(z)dz = R_0(z, f(z)) + \sum_{i=1}^{n} \lambda_i \cdot R_i(z, f(z))$$

- Davenport-Singer : if an inhomogeneous linear differential equation admits an elementary solution then it admits a (possibly different) elementary solution of the form

$$f(z) = P(z, \ln(S_1(z)), \dots, \ln(S_n(z))) \text{ with } P(z, -) \in \mathbb{C}(z)^{alg}[X_0, \dots, X_n]$$

Main drawback (compared to Liouville) : $S_i(z) \in \mathbb{C}(z)^{alg}$ as well as the coefficients of $P(z, -)$. This is nevertheless sharp.

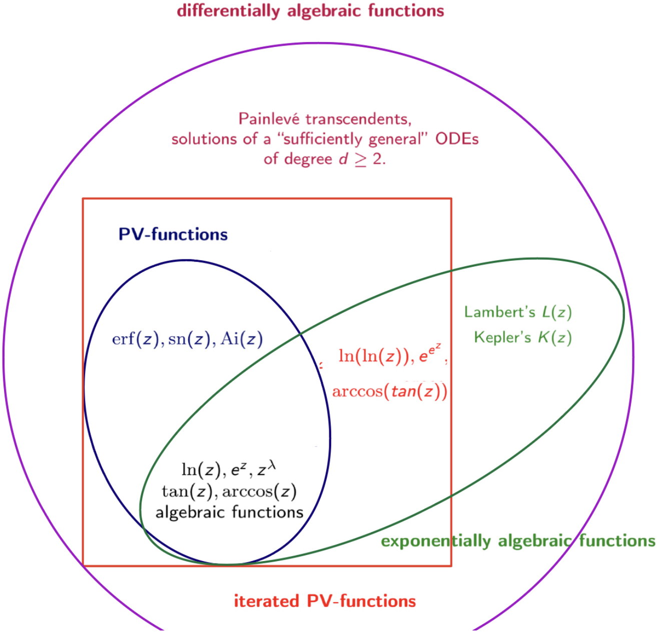
- J.- Kirby :

$$f(z) = H(z, \ln(u_i(z)), \exp(v_j(z)), (w_j(z))^{\lambda_j}, i, j, k = 1, \dots, n)$$

  for any (implicitly) elementary function $f(z)$ belonging to a Picard-Vessiot extension of $\mathbb{C}(z)^{alg}$.

Main drawback (compared to Singer-Davenport) : no information about the "shape" of the algebraic function H. This is nevertheless sharp.

A (partial) map of the universe of exponentially algebraic functions



Thank you for your attention !