On the problem of integration in finite terms and exponentially algebraic functions

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CIRM, September 30^{th} 2024

Introduction

Purpose of my talk. Describe a model-theoretic perspective on the problem of integration of finite terms.

- During the period between 1833 and 1841, Liouville presented a theory of integration in finite terms based on the notion of elementary function.
- As an application, Liouville studies functions of the differential calculus of the XIXth century such as

$$\mathrm{erf}(z) = \int e^{-z^2} dz, \mathrm{sn}(z) = \int \frac{dz}{\sqrt{z^3 + az + b}} \text{ and } \mathrm{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos(t^3/3 + tz) dt$$

  and shows that they can not be expressed "in a closed form" using the classical functions of calculus : exp, ln and the trigonometric functions.
- Since the second half of the XXth century, Liouville's theory have been revisited from several perspectives :
  - differential algebra (Ritt 1948, Rosenlicht 1970, Prelle-Singer 1983...)
  - topological Galois theory (Khovanskii 2015,...)
  - symbolic computations (Risch 1969, Davenport 1981,...)
  - transcendental number theory (Masser-Zannier 2020, Pila-Tsimerman 2022...)
- In model theory, elementary functions appear in the work of Van den Dries (1988) on the (strong) model completeness of the structure of restricted elementary functions :

$$\mathbb{R}^{\mathsf{RE}} = (\mathbb{R}, 0, 1, +, \times, \exp_{|[0,1]}, \sin_{|[0,\pi]}).$$

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Elementary functions in the sense of Liouville

An elementary function is a holomorphic function $f : U \subset \mathbb{C}^r \to \mathbb{C}$ which can be constructed in finitely many steps from the variables $z_1, \ldots, z_r$ using :

- arithmetic operations $(+, \times, -, /)$ involving possibly complex numbers,
- precomposition by the complex exponential and branches of the logarithm,
- extracting roots of (arbitrary) algebraic equations.

Examples.

$$f(z) = \frac{e^{\cos(z)}}{\sqrt{1+z^2}}(1 + \ln(1+z)) \text{ or } g(z_1, z_2) = \sqrt[n]{z_1 + z_1^2 \cdot \exp(z_2)}.$$

- **A word of warning from Ritt (1948).**

> There are, however, certain questions connected with the many-valued character of the elementary functions which could be pressed back behind the symbols in Liouville's time but which have since learned to assert their rights. Such matters are mulled over in the first chapter. The mulling is inescapable. It might be great fun to talk just as if the elementary functions were one-valued. I might even sound convincing to some readers; I certainly could not fool the functions.

- To resolve the ambiguity, it is often convenient to work with types in the theory DCF_0 rather than with holomorphic functions.

The problem of integration in finite terms

Liouville's problem. Given an elementary function $f(z)$ of one variable (say, defined on a simply connected domain),

(1) decide whether some/any antiderivative $\int f(z)dz$ is an elementary function

(2) compute $\int f(z)dz$ if it is an elementary function.

Some classical examples

- Rational functions. Using the partial fraction decomposition, one can always compute

$$\int P(z)/Q(z)dz = \underbrace{R(z)}_{\text{rational function}} + \overbrace{\sum_{\alpha \text{ root of Q}} \mathrm{Res}(P/Q, \alpha) \cdot \ln(z-\alpha)}^{\text{weighted sum of logs}}.$$

- Certain trigonometric integrals. Using integration by parts and substitutions,

$$\int \cos(z)^3 dz = \sin(z) - \sin(z)^3/3 \text{ but } \int \cos(z^2)dz = ??$$

- Quadratic formula. the antiderivative of "quadratic" algebraic functions can be computed using :

$$\int \frac{dz}{\sqrt{1-z^2}} = \arcsin(z) = -i \cdot \ln(iz + \sqrt{1-z^2}).$$

Mémoire sur l'intégration d'une classe de fonctions transcendantes (Crelle, 1833)

La première proposition dont je m'occupe est relative à l'intégrale $\int P dx$, dans laquelle P désigne une fonction algébrique $f(x, y, z, \ldots)$ de la variable indépendante x et de plusieurs fonctions de x, savoir y, z, etc. déterminées par un nombre égal d'équations différentielles telles que: $\frac{dy}{dx} =$ *une fonction algébrique de* x, y, z, *etc.;* $\frac{dz}{dx} =$ *une autre fonction algébrique de* x, y, z, *etc.;* et ainsi de suite. Je cherche quelle forme devra prendre la valeur de $\int P dx$, toutes les fois que cette valeur sera exprimable, sous forme finie, en fonction de x, y, z, etc., à l'aide des signes algébriques, exponentiels et logarithmiques. Et je démontre ainsi un théorème dont je vais écrire l'énoncé, et qui doit, à mon sens, être considéré comme fondamental dans la théorie de l'intégration des fonctions d'une seule variable.

Théorême. *Si l'intégrale* $\int P dx$ *est exprimable, sous forme finie, en fonction de* x, y, z *etc., à l'aide des signes algébriques, exponentiels et logarithmiques, il sera toujours permis de poser*

$$\int P dx = t + A \log u + B \log v + \ldots + C \log w;$$

$A, B, \ldots C$ *étant des constantes, et* $t, u, v, \ldots w$ *des fonctions algébriques de* x, y, z *etc.*

Liouville's theorem

Theorem (Liouville, 1833). Let $f(z)$ be an elementary function. If $\int f(z)dz$ is also elementary then it takes the special form

$$\int f(z)dz = \underbrace{R_0(z, f(z), f'(z), \ldots, f^{(n)}(z))}_{\text{rational function}} + \overbrace{\sum_{i=1}^{m} c_i \cdot \ln(R_i(z, f(z), \ldots, f^{(n)}(z))}^{\text{weighted sum of logs}}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ and $R_0, \ldots, R_m \in \mathbb{C}(X_0, \ldots, X_{n+1})$.

- In particular for $f(z) = e^{-z^2}$, $f'(z) = -2zf(z)$ and Liouville reaches :

$$\int e^{-z^2} dz = R_0(z, e^{-z^2}) + \sum_{i=1}^{m} c_i \cdot \ln(R_i(z, e^{-z^2})),$$

takes the derivative and shows that it does not admit any solution.

- Developing a (sophisticated) structure theory based on Liouville's theorem, Risch (1970+) shows that the problem of integration in finite terms is decidable. It is considered as a milestone in the development of mathematics.

> I would like to express my indebtedness to Professor James Ax for his advice and suggestions during the writing of this note. I would also like to thank Professor Goro Shimura for information concerning good reduction.

- The complete (published) proof of decidability only appears in a recent volume (2022) edited by Singer and Raab.

How to improve on Liouville's theory ?

Objective. Replace in Liouville's theory

$$\left\{ \begin{array}{l} \text{Liouville's class of} \\ \text{elementary functions} \end{array} \right. \rightsquigarrow \left\{ \begin{array}{l} \text{holomorphic functions locally definable in} \\ \mathbb{R}^{\mathbf{RE}} = (\mathbb{R}, 0, 1, +, \times, \exp_{|[0,1]}, \sin_{|[0,\pi]}). \end{array} \right.$$

Motivation

- Elementary functions vs implicitly defined elementary functions. The larger class of functions is stable under the implicit function theorem.

Liouville (1837) shows that the Lambert W-function given by

$$W(z) = \sum_{n=0}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \text{ solution of } W(z) \cdot e^{W(z)} = z$$

is not an elementary function.

- In a slightly more "modern" context,

$$\left\{ \begin{array}{l} \text{class of} \\ \text{Pfaffian functions} \end{array} \right. \rightsquigarrow \left\{ \begin{array}{l} \text{holomorphic functions locally definable in} \\ \mathbb{R}^{\mathbf{Pfaff}} = (\mathbb{R}, (\text{rest. Pfaffian functions })) \end{array} \right.$$

Freitag shows that the j-function can not be integrated by Pfaffian functions but Jones and Speissegger show that it is locally definable in $\mathbb{R}^{\mathbf{Pfaff}}$.

- Account for additional formulas from calculus.

$$\int \frac{W(z)+1}{z} dz = W(z)^2/2 + W(z) + \ln(z) \text{ but } \int \frac{W(z)}{z^2} dz = ??$$

The short answer

Theorem (J., Kirby). Let $f(z)$ be a holomorphic function locally definable in $\mathbb{R}^{\mathbf{RE}}$. If $\int f(z)dz$ is also locally definable in $\mathbb{R}^{\mathbf{RE}}$ then it takes the special form

$$\int f(z)dz = \underbrace{R_0(z, f(z), f'(z), \ldots, f^{(n)}(z))}_{\text{rational function}} + \overbrace{\sum_{i=1}^{m} c_i \cdot \ln(R_i(z, f(z), \ldots, f^{(n)}(z))}^{\text{weighted sum of logs}}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ and $R_0, \ldots, R_m \in \mathbb{C}(X_0, \ldots, X_{n+1})$.
In particular, if $f(z)$ is an elementary function then $\int f(z)dz$ is locally definable in $\mathbb{R}^{\mathbf{RE}}$ if and only if it is elementary.

Remark. In the case where $f(z)$ is an algebraic function, an equivalent form of this statement was already known to... Abel (1802-1829) and hence (in some sense) this statement even precedes Liouville's theorem !

> Another generalization of Liouville's Theorem stated by Abel is: if the integral of an algebraic function can be expressed in terms of implicitly or explicitly defined elementary functions then it is elementary and so satisfies the conclusion of Liouville's Theorem. Lützen remarks that it is not clear what Abel meant by implicit elementary functions as no proof survives, but Ritt [Rit48, pp. 71–98] and Risch [Ris76] proved precise versions of this result.

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Exponential algebraicity for holomorphic functions

- The concept of exponential algebraicity goes back to the work of Macintyre and Zilber on exponential algebra. To any (full) exponential field $(K, \exp)$, they associate a pregeometry
$$\mathrm{ecl}_K : \mathcal{P}(K) \to \mathcal{P}(K)$$
One of the main goal is to get a notion applicable to the study of "numbers" i.e. to the structure $(\mathbb{C}, \exp)$.
- Instead, we are interested in a notion of exponential algebraicity applicable to the study of holomorphic and meromorphic functions only.

Two important differences.

(1) "functions behave better than numbers" : this pregeometry comes as the forking pregeometry of a regular type of rank ω in an ω-stable theory.

(2) "functions can be restricted and glued together" : the notion should (at least) be preserved under restriction of function to smaller open sets.

To account for these differences, we work inside a model $(\mathcal{U}, \partial) \models \mathrm{DCF}_0$ with field of constants $\mathbb{C}$ and work with a blurred version of an exponential map given as a subgroup

$$\Gamma_{exp} := \{(x, y) \in \mathcal{U} \times \mathcal{U}^* \mid \partial(x) = \frac{\partial(y)}{y}\} \subset (\mathcal{U}, +) \times (\mathcal{U}^*, \times).$$

In a functional framework, similar uses of exponential algebraicity for holomorphic functions goes back to the work of Wilkie (2008) and Jones, Kirby, Servi (2014).

Blurred exponential fields

Definition. Let K/C be an extension of algebraically closed fields of characteristic zero and $\Gamma \subset K \times K^*$. We say that (K, C, Γ) is a exponential field blurred by C (or simply a blurred exponential field) if

(i) Γ is a divisible subgroup of $(K, +) \times (K^*, \times)$,

(ii) $\Gamma \cap (K \times C^*) = C$ and $\Gamma \cap (C \times K^*) = C^*$.

Comments.

- Blurred exponential fields are studied in the language $\mathcal{L}_\Gamma = \mathcal{L}_{rings} \cup \{\Gamma\}$ and are a particular case of Γ-fields in the sense of Bays and Kirby (2015).
- If (K, ∂) is an algebraically closed differential field with field of constants C and

$$\Gamma_\partial := \{(x, y) \in \mathcal{U} \times \mathcal{U}^* \mid \partial(x) = \frac{\partial(y)}{y}\} \subset (K, +) \times (K^*, \times)$$

  then (K, C, Γ_∂) is an exponential field blurred by C.
- There is a “distinguished” complete $\mathcal{L}_\Gamma$-theory :

$$\mathrm{BE}_0 := \mathrm{Th}_{\mathcal{L}_\Gamma}(K, C, \Gamma_\partial) \text{ for } (K, \partial) \models \mathrm{DCF}_0$$

  which inherits as a reduct the good tameness properties of DCF_0.
- A first-order axiomatization of BE_0 based on the Ax-Schanuel Theorem and existential closedness axioms has been described by Kirby (2007).

A paradox : passing to a reduct to extract more information...

The problem of integration in finite terms requires to "measure" the difference between

$$f_1(z) = \arcsin(z) = \int \frac{dz}{\sqrt{1-z^2}} \text{ and } f_2(z) = \mathrm{sn}(z) = \int \frac{dz}{\sqrt{z^3+az+b}}.$$

Working inside DCF_0. Set $p_i = \mathrm{tp}^{\mathrm{DCF}_0}(f_i(z)/\mathbb{C}(z)^{alg})$ for $i = 1, 2$.

- p_1 and p_2 are both strongly minimal types.
- p_1 and p_2 are both internal to the constants,
- p_1 and p_2 both have the additive group for Galois group,
- ... but the two equations (E_1) and (E_2) isolating the types

$$(E_1) : y' = 1/\sqrt{1-z^2} \text{ and } (E_2) : y' = 1/\sqrt{z^3+az+b}$$

are not "gauge equivalent".

Working inside BE_0. Set $q_i = \mathrm{tp}^{\mathrm{BE}_0}(f_i(z)/\mathbb{C}(z)^{alg})$ for $i = 1, 2$.

- q_1 is strongly minimal and q_2 is the unique type of rank ω. (to be explained !)

Conclusion of this observation. We will study the PIFT using the continuous map

$$\mathrm{red} : S^{\mathrm{DCF}_0}(\mathbb{C}(z)^{alg}) \to S^{\mathrm{BE}_0}(\mathbb{C}(z)^{alg})$$

associated with passing to the reduct $\mathrm{DCF}_0 \to \mathrm{BE}_0$.

Description of the types of finite Morley rank in BE_0

Question. When does $\mathrm{tp}^{\mathrm{BE}_0}(\overline{g(z)}/\mathbb{C}(z)^{alg})$ have finite Morley rank ?

The Ax-Schanuel Theorem (1971). Let $f_1, \ldots, f_n : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic functions. Then

$$\mathrm{td}(f_1(z), \cdots, f_n(z), e^{f_1(z)}, \ldots, e^{f_n(z)}/\mathbb{C}(z)) \geq n$$

provided the functions $f_1, \ldots, f_n$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$.

Characterization of the types of fMR in BE_0. Let $g_1, \ldots, g_n : U \subset \mathbb{C} \to \mathbb{C}$ be a tuple of holomorphic functions. The following are equivalent :

(i) the type $\mathrm{tp}^{\mathrm{BE}_0}(\overline{g(z)}/\mathbb{C}(z)^{alg})$ has finite Morley rank,

(ii) There exists N holomorphic functions $f_1, \ldots, f_N : V \subset U \to \mathbb{C}$, $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ satisfying the equality case in the Ax-Schanuel inequality that is

$$\mathrm{td}(f_1(z), \cdots, f_N(z), e^{f_1(z)}, \ldots, e^{f_N(z)}/\mathbb{C}(z)) = N$$

such that

$$g_i(z) = H_i(z, f_1(z), \ldots, f_N(z), e^{f_1(z)}, \ldots, e^{f_N(z)}), i = 1, \ldots, n$$

where $H_1, \ldots, H_n \in \mathbb{C}(X_0, \ldots, X_{2N})^{alg}$.

Comment. The natural number N appearing in (ii) is then an upper bound for the Morley rank of $\mathrm{tp}^{\mathrm{BE}_0}(\overline{g(z)}/\mathbb{C}(z)^{alg})$ but this is not sharp.

Description of the types of finite Morley rank in BE_0

Question. When does $\mathrm{tp}^{\mathrm{BE}_0}(\overline{g(z)}/\mathbb{C}(z)^{alg}) = \mathrm{tp}^{\mathrm{BE}_0}(\overline{h(z)}/\mathbb{C}(z)^{alg})$ assuming that both have finite Morley rank ?

Exponential hull. Assuming that N is chosen minimal such that (ii) holds in the previous characterization then set

$$\mathrm{Hull}(\mathbb{C}(z, \overline{g}(z))) := \mathbb{C}(z, f_1(z), \ldots, f_N(z), e^{f_1(z)}, \ldots, e^{f_N(z)})^{alg}.$$

This hull does not depend on the choices of the functions $f_1(z), \ldots, f_N(z))$ realizing (ii).

Characterization of the types of fMR in BE_0. Let $g_1, \ldots, g_n, h_1, \ldots, h_n : U \subset \mathbb{C} \to \mathbb{C}$ be two tuples of holomorphic functions of fMR over $\mathbb{C}(z)^{alg}$. The following are equivalent :

(i) $\mathrm{tp}^{\mathrm{BE}_0}(\overline{g(z)}/\mathbb{C}(z)^{alg}) = \mathrm{tp}^{\mathrm{BE}_0}(\overline{h(z)}/\mathbb{C}(z)^{alg})$,

(ii) There exists an isomorphism of $\mathcal{L}_\Gamma$-structures

$$\phi : \mathrm{Hull}(\mathbb{C}(z, \overline{g}(z))) \to \mathrm{Hull}(\mathbb{C}(z, \overline{h}(z)))$$

fixing $\mathbb{C}(z)^{alg}$ and sending $\overline{g}(z)$ to $\overline{h}(z)$.

Description of red : $S^{\mathrm{DCF}_0}(\mathbb{C}(z)^{alg}) \to S^{\mathrm{BE}_0}(\mathbb{C}(z)^{alg})$

Theorem (Wilkie 2008, Kirby 2007-2010). Let $p \in S_n^{\mathrm{DCF}_0}(\mathbb{C}(z)^{alg})$ be a type which is weakly orthogonal to the constants. The following are equivalent :

(i) (realization in $\mathbb{R}^{\mathsf{RE}}$) there exists n holomorphic functions definable in

$$\mathbb{R}^{\mathsf{RE}} = (\mathbb{R}, 0, 1, +, \times, \exp_{|[0,1]}, \sin_{|[0,\pi]})$$

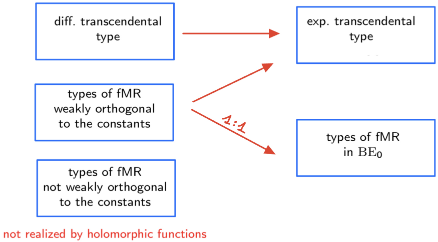
realizing the type p.

(ii) (types of fMR in the reduct) $\mathrm{red}(p)$ has finite Morley rank in BE_0.

(iii) (faithful reduction) $\mathrm{red}(p) \vdash p$ in DCF_0.

Under these conditions, we say that p is an exponentially algebraic type.

At the level of 1-types...



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Definable Galois theory in BE_0

- Definable Galois theory is a geometric stability device introduced by Poizat and developed by Pillay during the nineties in relation with the notion of internality (in the sense of geometric stability theory).
- In the ω-stable theories DCF_0 and more recently CCM, its study provides key insights about the fine structure of the type spaces. Similarly for BE_0, ...

Theorem 1. (J., Kirby) Let $p \in S_1^{\mathrm{DCF}_0}(\mathbb{C}(z)^{alg})$ be an exponentially algebraic type.

(i) (Zilber's trichotomy) up to nonorthogonality, there is a unique minimal nonlocally modular type in the theory BE_0 : the generic type of the pure algebraically closed field $\mathcal{C}$ of "constants".

(ii) (compatibility between the Galois theories) the type p is internal to the constants in DCF_0 if and only if $\mathrm{red}(p)$ is internal to the "constants" in BE_0 and

$$\mathrm{Aut}^{\mathrm{DCF}_0}(p/\mathcal{C}) \simeq \mathrm{Aut}^{\mathrm{BE}_0}(\mathrm{red}(p)/\mathcal{C}).$$

(iii) (inverse Galois problem) if p is internal to the constants then the binding group $\mathrm{Aut}^{\mathrm{DCF}_0}(p/\mathcal{C})$ is of the form

$$\mathrm{Aut}^{\mathrm{DCF}_0}(p/\mathcal{C}) \simeq \mathbb{G}_a^k(\mathcal{C}) \times \mathbb{G}_m^l(\mathcal{C}).$$

Some Liouville calculus based on Theorem 1

Corollary. The three functions

$$\mathrm{erf}(z) = \int e^{-z^2} dz,\ \mathrm{sn}(z) = \int \frac{dz}{\sqrt{z^3 + az + b}} \text{ and } \mathrm{Ai}(z) = \int_0^\infty \cos(u^3/3 + uz) du$$

are not locally definable in $\mathbb{R}^{\mathsf{RE}}$.

Proof

- Their types over $\mathbb{C}(z)^{alg}$ are isolated respectively by the formulas

$$\underbrace{y'' - 2z \cdot y' = 0, y' \neq 0}_{\mathrm{Aff}_2(\mathbb{C})},\ \underbrace{y' = \frac{1}{\sqrt{z^3 + az + b}}}_{\mathrm{G}_a(\mathbb{C})} \text{ and } \underbrace{y'' - z \cdot y = 0, y \neq 0}_{\mathrm{SL}_2(\mathbb{C})}.$$

- For $\mathrm{erf}(z)$ and $\mathrm{Ai}(z)$, the Galois-theoretic information is conclusive. For sn(z), any local compositional inverse of $\mathrm{sn}(z)$ satisfies the elliptic equation

$$\underbrace{(y')^2 = y^3 + ay + b}_{\text{an elliptic curve } E(\mathbb{C})}.$$

Using the same "trick" of considering local compositional inverses :

Corollary. The j-function is not locally definable in $\mathbb{R}^{\mathsf{RE}}$.

Integrability by elementary functions vs integrability in $\mathbb{R}^{\mathbf{RE}}$

Theorem 2. (J., Kirby) Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be an holomorphic function. The function f is an elementary function in the sense of Liouville if and only if

(a) the function f is locally definable in $\mathbb{R}^{\mathbf{RE}}$,

(b) $\mathrm{tp}^{\mathrm{DCF}_0}(f(z)/\mathbb{C}(z)^{alg})$ is analyzable in the constants.

Application to more Liouville calculus...

- Risch's decision procedure (or any practical implementation of it on a computer software) can be used to decide whether the antiderivative $\int f(z)dz$ of an elementary function $f(z)$ is locally definable in $\mathbb{R}^{\mathbf{RE}}$:

$$\mathrm{tp}^{\mathrm{DCF}_0}(f(z)/\mathbb{C}(z)^{alg}), \mathrm{tp}^{\mathrm{DCF}_0}(\int f(z)dz/\mathbb{C}(z, f(z))^{alg}) \text{ are both analyzable in } \mathcal{C}.$$

- Similarly, any solution $f(z)$ locally definable in $\mathbb{R}^{\mathbf{RE}}$ of a linear differential equation

$$y^{(n)} + a_{n-1}(z) \cdot y^{(n-1)} + \cdots a_0(z) \cdot y(z) = b(z)$$

whose coefficients are elementary functions is in fact elementary.

- On the other hand, the Lambert W-function

$$W(z) = \sum_{n=0}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \text{ solution of } z(1+W)\frac{dW}{dz} = W$$

satisfies that $\mathrm{tp}^{\mathrm{DCF}_0}(W(z)/\mathbb{C}(z))$ is strongly minimal and geometrically trivial.

A generalization of Liouville's Theorem

Theorem 3 (J., Kirby). Let $f(z)$ be a holomorphic function locally definable in $\mathbb{R}^{\mathsf{RE}}$. If $\int f(z)dz$ is also locally definable in $\mathbb{R}^{\mathsf{RE}}$ then it takes the special form

$$\int f(z)dz = \underbrace{R_0(z, f(z), f'(z), \ldots, f^{(n)}(z))}_{\text{rational function}} + \overbrace{\sum_{i=1}^{m} c_i \cdot \ln(R_i(z, f(z), \ldots, f^{(n)}(z))}^{\text{weighted sum of logs}}$$

where $c_1, \ldots, c_m \in \mathbb{C}$ and $R_0, \ldots, R_m \in \mathbb{C}(X_0, \ldots, X_{n+1})$.

Comments.

- in the case where $f(z)$ is an elementary function, then one apply the strategy described by Singer to obtain Abel's generalization of Liouville theorem :

  Theorem 2 + original Liouville's theorem $\Rightarrow$ Theorem 3.

- in the general case, we adapt Rosenlicht's differential algebraic proof of Liouville's theorem to this set-up using the (concrete) description of types of fMR in BE_0.

Three open questions

- (Minimal types in BE_0) Using Zilber's notion of perfectly rotund subvariety of $(\mathbb{G}_a \times \mathbb{G}_m)^n$, it is possible to show the existence for every $n \geq 1$ of minimal exponentially algebraic type $p \in S_1^{\mathrm{DCF}_0}(\mathbb{C}(z)^{alg}$ of order n.

Question. Are all minimal locally modular exponentially algebraic types ω-categorical ? How to show that the generic types of the Manin Kernels are not exponentially algebraic ?

- (Decidability for n-types in BE_0) Considering several functions instead of a single one, an analogue of the PIFT is to compute the exponential transcendence degree :

$$(f_1(z), \ldots, f_n(z)) \mapsto \mathrm{etd}(f_1(z), \ldots, f_n(z)/\mathbb{C}(z)^{alg}) \in \{0, \ldots, n\}$$

  For example, we can show that $\mathrm{etd}(\mathrm{erf}(z), \mathrm{sn}(z), \mathrm{Ai}(z)/\mathbb{C}(z)^{alg}) = 3$.

Question. Is there a decision procedure which given n algebraic functions $f_1(z), \ldots, f_n(z)$ computes the exp. trans. degree of $\int f_1(z)dz, \ldots, \int f_n(z)dz$ over $\mathbb{C}(z)^{alg}$?

- (elliptic and abelian functions) The direct analogue of the Kirby-Wilkie Theorem holds true taking into account the exponential maps

$$\exp : LA \to A \text{ of all complex semi-abelian varieties } A.$$

Question. Is it possible to prove analogues of Theorem 1 and Theorem 3 including the exponential maps of some/all complex semi-abelian varieties ?

Thank you for your attention !